Reachability Problems for Hybrid Automata

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based on joint works with
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Content

- Motivations: reactive embedded and hybrid systems
- **Classes** of hybrid automata
- **Symbolic semi-**algorithm for reachability
- Reachability problem: **decidability** frontier
- **Approximate** reachability
- **Time-bounded** reachability

Reactive and hybrid systems

Reactive systems maintain a continuous interaction with their environment

- non-terminating
- respect/enforce real-time properties
- cope with concurrency
- embedded in complex-continuous-critical env

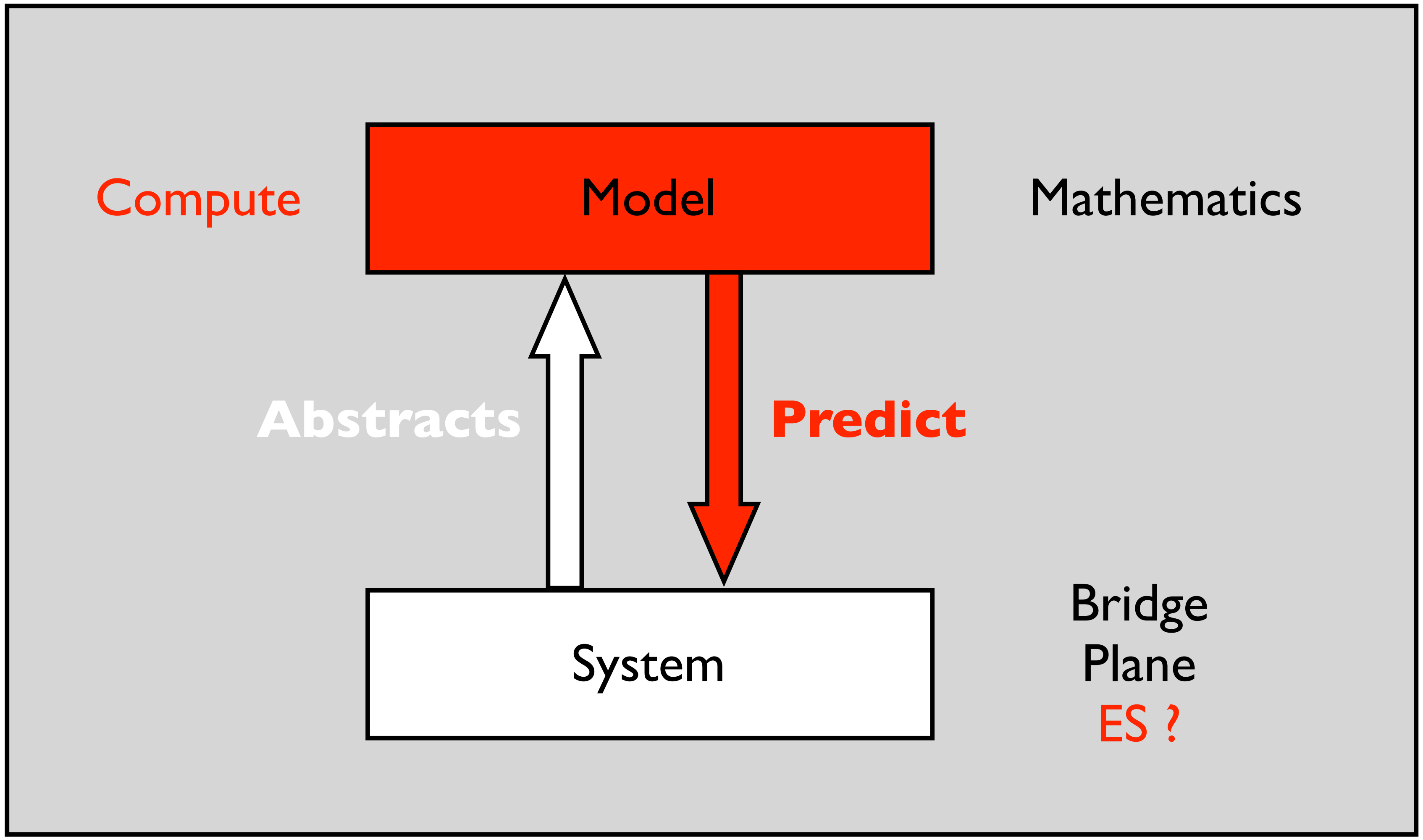
→ difficult to develop correctly

300 horses power

100 processors

Is the software correct ?

How to cope with complexity



Hybrid automata

Mixing discrete-continuous evolutions

- Finite state automata to model (discrete) reactive systems
- Differential equations to model continuous environments
- Hybrid automata: combine the two
  - finite automata + continuous variables
  - discrete transitions + differential equations

Example

- Three environment components:

  - A tank containing water;
  - A gas burner that can be turn on or off;
  - A digital thermometer that monitors the temperature within the tank.

  and a controller

- We want to design a controller strategy that maintains the temperature within an interval of safe temperatures.

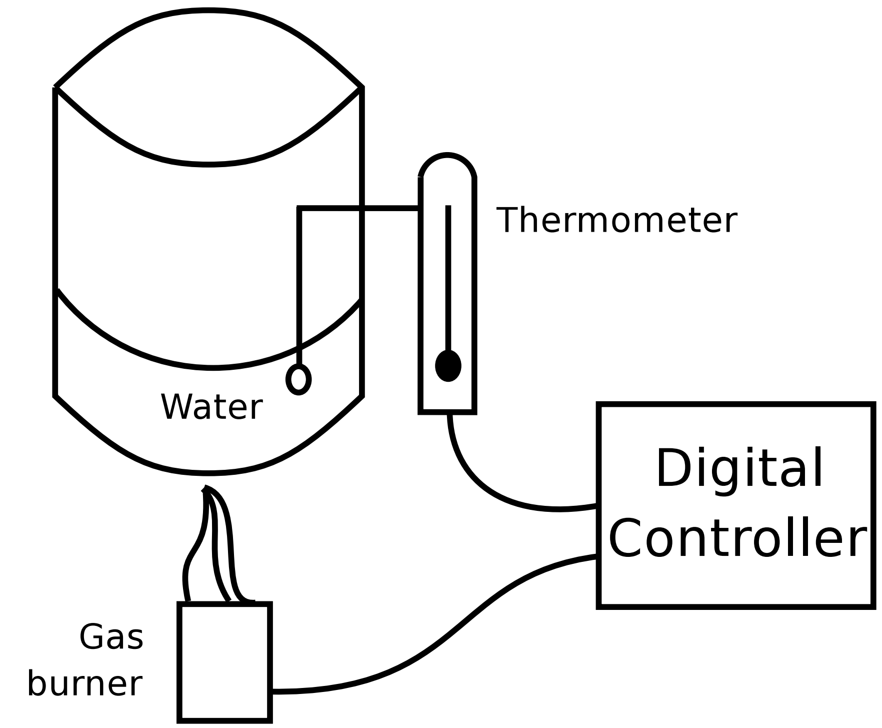


Fig. 1. Our running example

Continuous part

- Behavior of the temperature in the tank

  -Mode **OFF**: $\mathbf{x(t) = I\, e^{-Kt}}$, i.e. $\dot{x} = -Kx$

  -Mode **ON**: $\mathbf{x(t) = I\, e^{-Kt} + h\,(1 - e^{-Kt})}$, i.e. $\dot{x} = K(h-x)$

  I=initial temperature of the water
  K=constant (nature of the tank)
  h=constant (power gas burner)
  t=time.

- **ON** and **OFF**=modes of the tank evolution

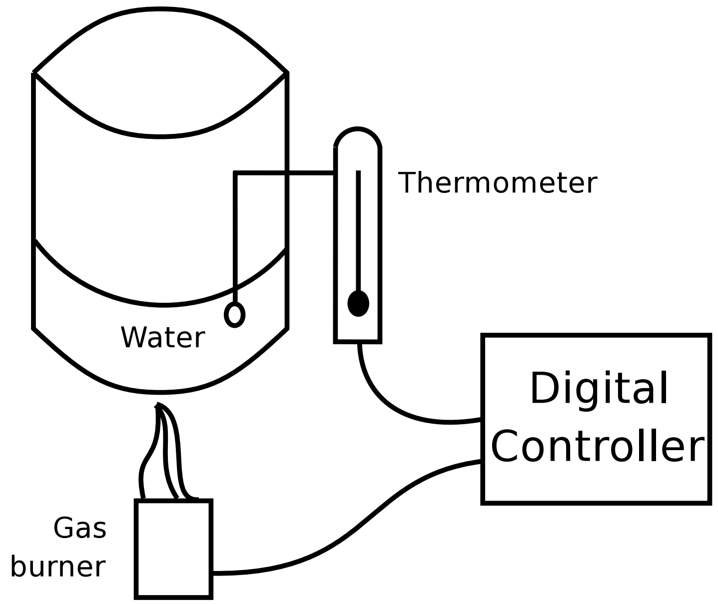
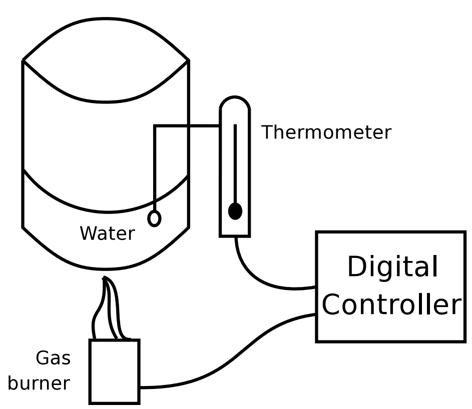


Fig. 1. Our running example

Evolution of the temperature

Mode changes

Continuous Evolutions



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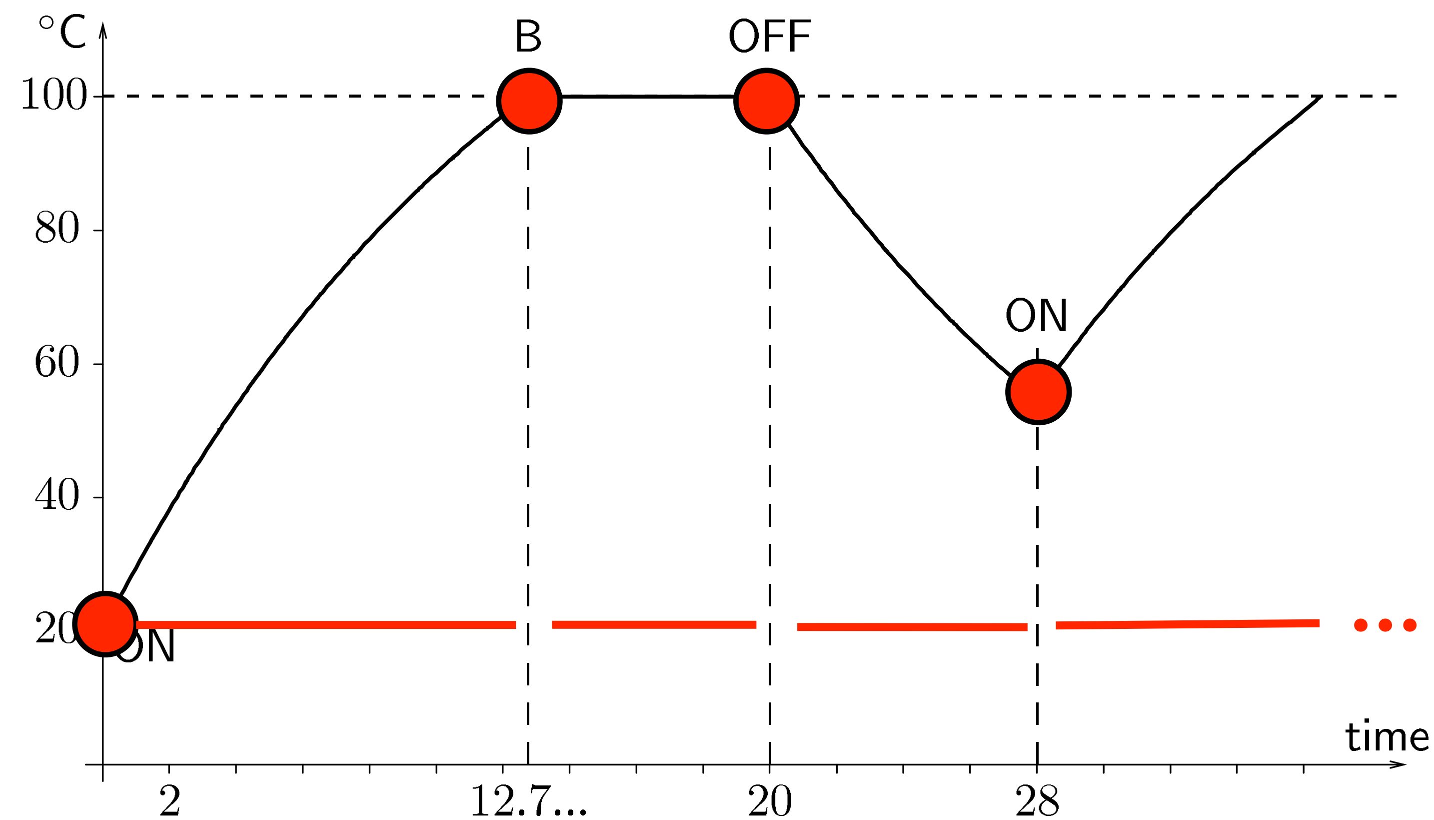
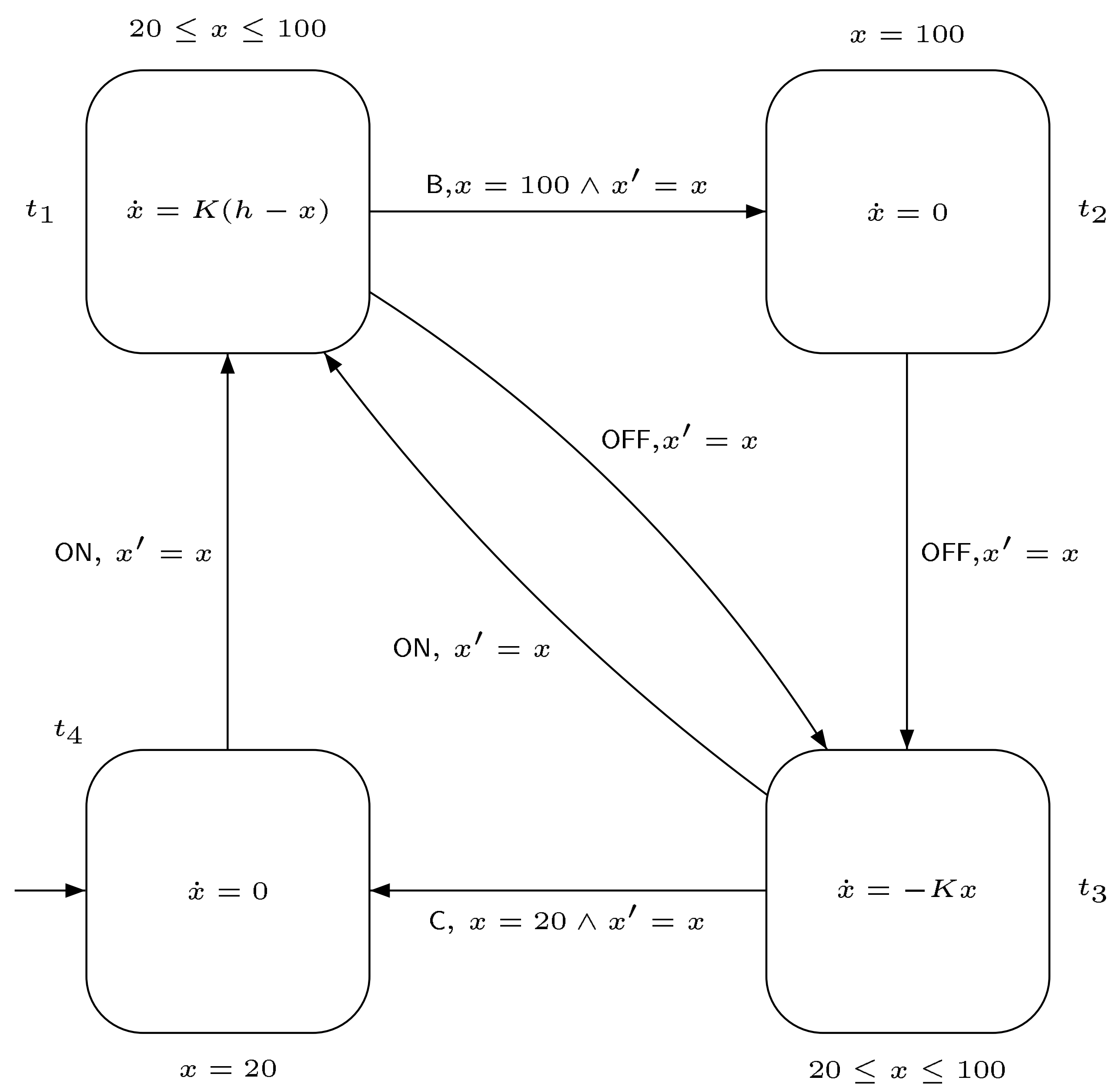


Fig. 2. One possible behavior of the tank

Evolution of the temp. is **not** purely continuous. It depends on the mode **ON** and **OFF** for example, and that it is below 100° or not.

An HA for the tank



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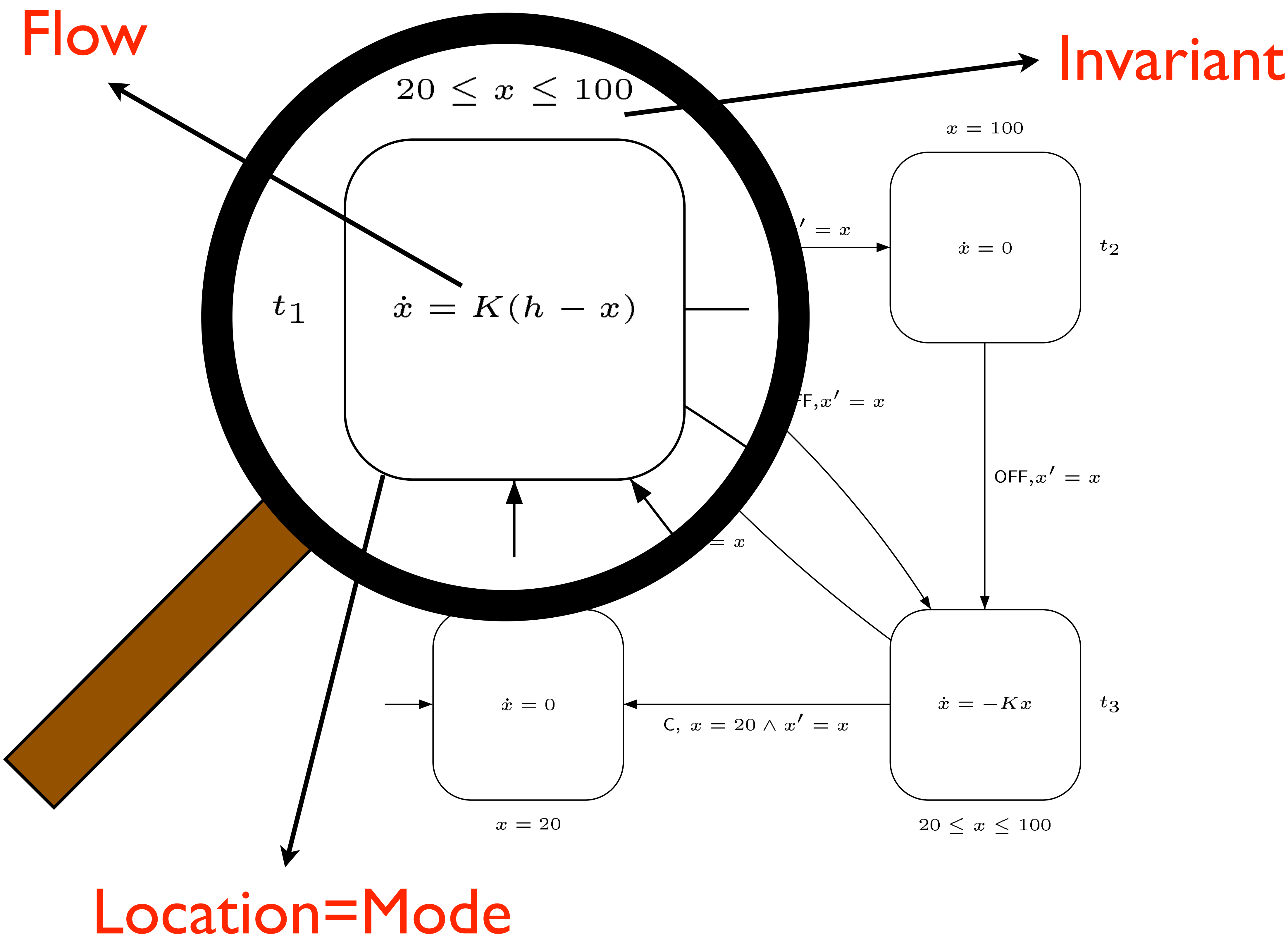

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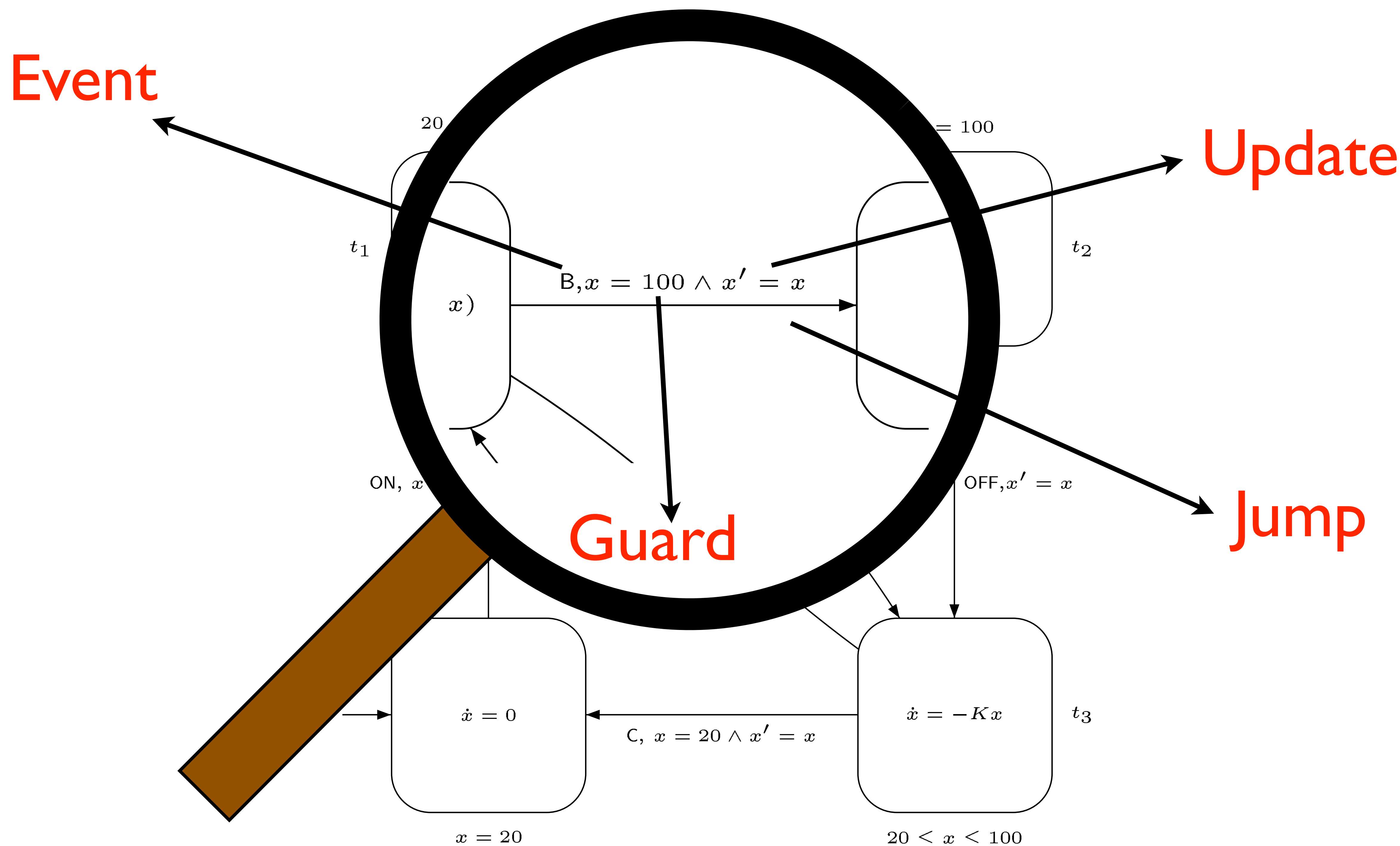

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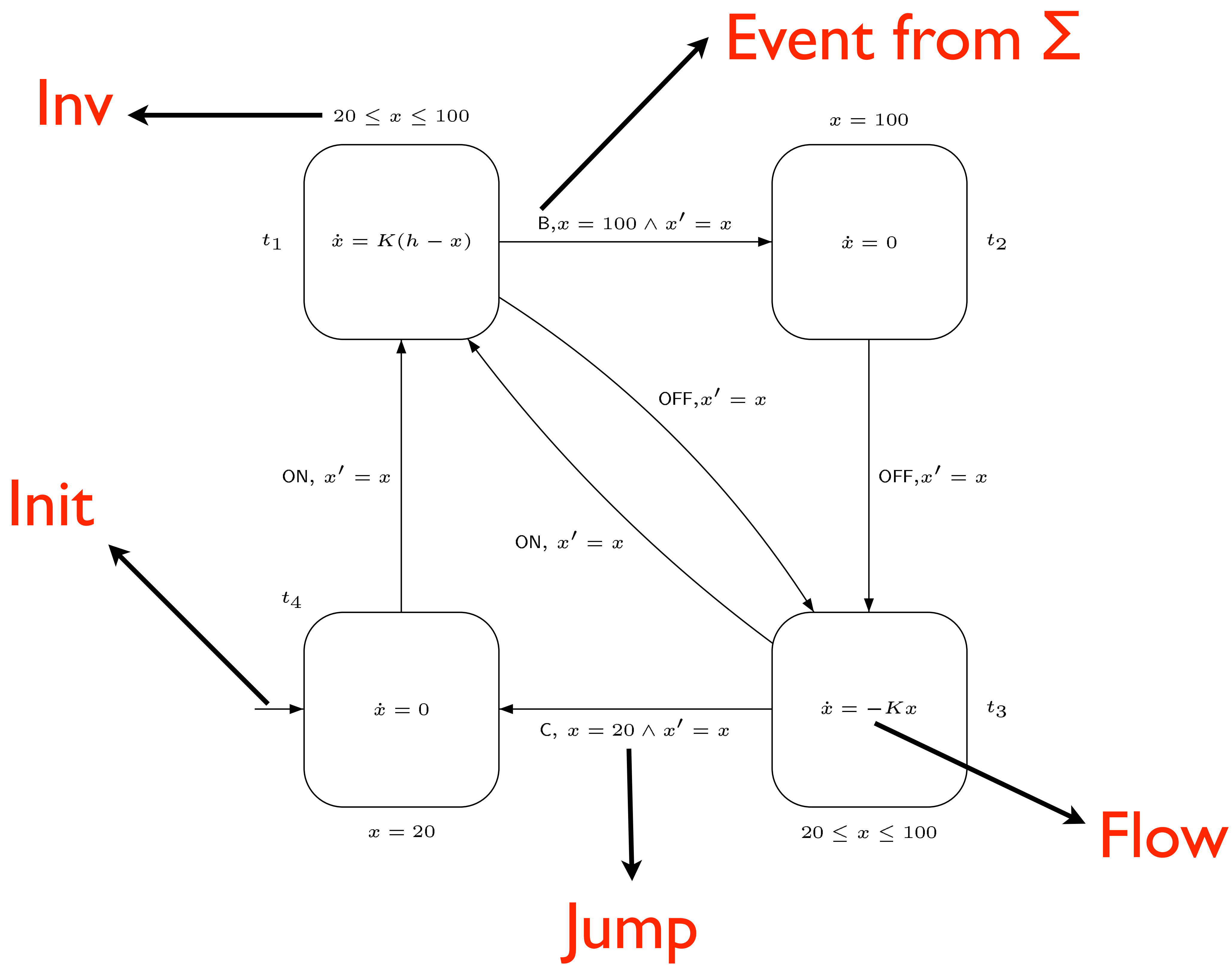


Hybrid automata - Syntax

Definition

H=(Loc,Σ,Edge,X,Init,Inv,Flow,Jump), where:

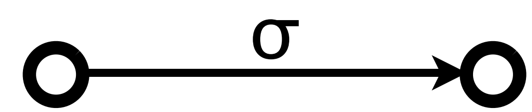
- **Loc** is a finite set $\{l_1,l_2,\ldots,l_n\}$ of (control locations) modeling control modes
- **Σ** is a finite set of event names
- **Edge** ⊆ Loc × Σ × Loc is a finite set of labelled edges modeling discrete changes between control modes
- **X** is a finite set $\{x_1,x_2,\ldots,x_m\}$ of real-valued variables.
  - We write $\dot{X}=\{\dot{x}_1,\dot{x}_2,\ldots,\dot{x}_m\}$ for the dotted variables and
  - $X'=\{x'_1,x'_2,\ldots,x'_m\}$ for the primed variables
- **Init(X)**, **Inv(X)**, and **Flow(X,$\dot{X}$)** are predicates associated to locations
- **Jump(X,X')** is a function that assigns a predicate to each labelled edge

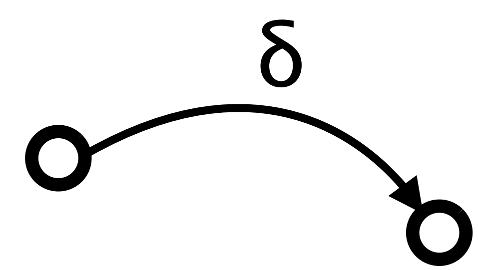
TTS of a HA

- Let H=(Loc,Σ,Edge,X,Init,Inv,Flow,Jump) be a HA.
- Its associated **T**imed **T**ransition **S**ystem ⟦H⟧=(S,S_0,Σ,→) is defined as follows:
  - S is the set of pairs (l,v) where l∈Loc and v∈⟦Inv(l)⟧;
  - S_0 is the subset of pairs (l,v)∈S such that v∈⟦Init(l)⟧;

Timed transition system of a HA

Transition relation





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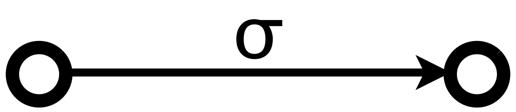
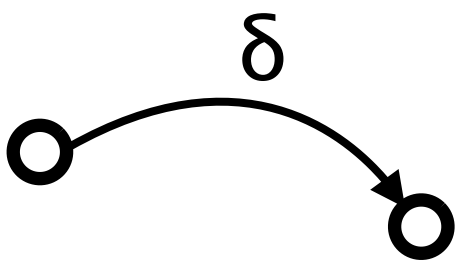


- discrete steps:
  for each edge $e=(l,\sigma,l')\in E$, we have $(l,v)\rightarrow_{\sigma}(l',v')$
  if $(l,v)\in S$, $(l',v')\in S$ and $(v,v')\in[\![Jump(e)]\!]$;



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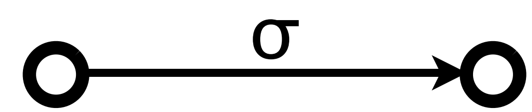

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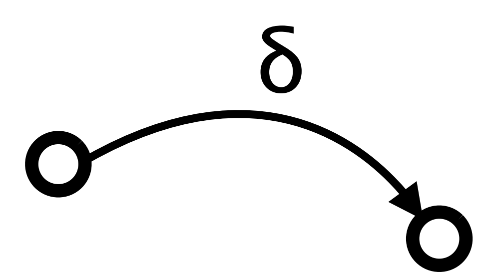
  if $(l,v)\in S$, $(l',v')\in S$ and $(v,v')\in[\![Jump(e)]\!]$;
- continuous steps: for each $\delta\in\mathbb{R}\geq 0$, we have $(l,v)\rightarrow_{\delta}(l',v')$
  if $(l,v)\in S$, $(l',v')\in S$, $l=l'$,
  and there exists a *differentiable function* $f:[0,\delta]\rightarrow\mathbb{R}^m$,
  with derivative $\dot{f}(0,\delta)\rightarrow\mathbb{R}^m$
  such that :
  1) $f(0)=v$,
  2) $f(\delta)=v'$ and
  3)for all $\varepsilon\in(0,\delta)$, both



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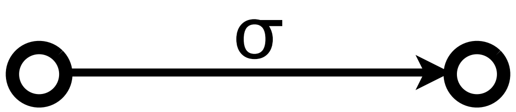

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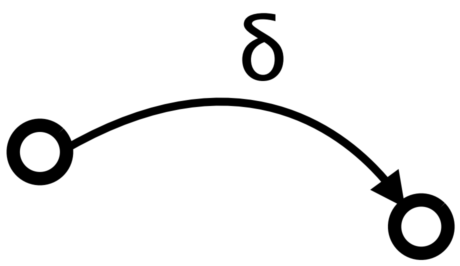
  - $f(\varepsilon)\in[\![Inv(l)]\!]$ and



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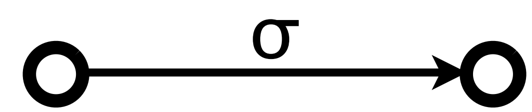

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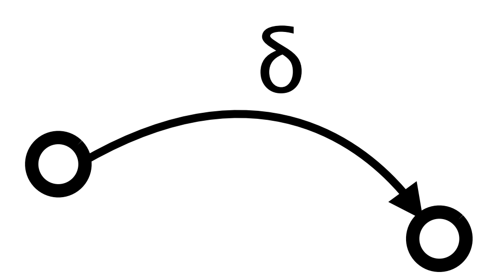
  - $f(\varepsilon)\in[\![Inv(l)]\!]$ and
  - $(f(\varepsilon),\dot{f}(\varepsilon))\in[\![Flow(l)]\!]$.



Reachability

- Let $Path_F(S_0)$=set of finite paths starting from a state in S_0
- Let $T=(S,S_0,\Sigma,\rightarrow)$ be a TTS
  Let $\lambda=s_0\tau_0s_1\tau_1\ldots s_n \in Path_F(T)$
  State(λ) denotes the set of states that appear along λ
- We say that a path λ reaches a state s if $s \in State(\lambda)$
- We say that s is reachable in T if $s \in \cup_{\lambda \in PathF(T)} State(\lambda)$
- Reach(T) denotes the set of states reachable in T

Safety and reachability

- A set of state $R \subseteq S$ is called a region.
- A region R is reachable in T iff $R \cap Reach(T) \neq \emptyset$.
- The rechability problem associated to a TTS T and a region R asks if $R \cap Reach(T) \neq \emptyset$.
- The safety problem associated to a TTS T and a region R asks if $Reach(T) \subseteq R$.
- Those two problems are dual in the following formal sense:

  Let R be a region and $R' = S \setminus R$.

$Reach(T) \subseteq R$ iff $R' \cap Reach(T) = \emptyset$.

Classes of Hybrid Automata

Classes of HA

Linear HA

- **Linear** flow constraints:
  Lin($X^\bullet$), **ex:** $x^\bullet = y^\bullet + 3$

- **Linear** guards and updates:
  Lin(X)→Lin(X,X'),
  ex: $x+y<1 \rightarrow x' := y+2$

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Rectangular HA

- **Rectangular** flow constraints:
  Rect($X^\bullet$), **ex**: $x^\bullet\in[1,2]\wedge y^\bullet\in[2,5]$
- **Rectangular** guards-updates:
  Rect(X)→Rect(X')
  ex: $x\in[2,5]\rightarrow x'\in[5,7]$

Classes of HA

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- **Linear** flow constraints: Lin($X^\bullet$), **ex:** $x^\bullet = y^\bullet + 3$
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Affine HA

- **Affine** flow constraints: Aff($X, X^\bullet$), ex: $x^\bullet = 2x+3y$
- **Linear** guards and updates: Lin(X)→Lin(X,X'), **ex:** $x+y<1 \rightarrow x' := y+2$

Rectangular HA

- **Rectangular** flow constraints: Rect($X^\bullet$), **ex**: $x^\bullet \in [1,2] \wedge y^\bullet \in [2,5]$
- **Rectangular** guards-updates: Rect(X)→Rect(X') **ex**: $x \in [2,5] \rightarrow x' \in [5,7]$

Classes of HA

Linear HA

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Affine HA

- **Affine** flow constraints: $Aff(X,X^{\bullet})$, ex: $x^{\bullet}=2x+3y$
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Rectangular HA

- **Rectangular** flow constraints: $Rect(X^{\bullet})$, **ex**: $x^{\bullet} \in [1,2] \wedge y^{\bullet} \in [2,5]$
- **Rectangular** guards-updates: $Rect(X) \rightarrow Rect(X')$ **ex**: $x \in [2,5] \rightarrow x' \in [5,7]$

O-minimal HA

- Use of **O-minimal theory**
- **Strong resets**: all variables are reset during any mode change

Symbolic Semi-Algorithm for RHA/LHA

Effective procedure for Post in RHA

Effective procedure for Post in RHA

- A linear term over X is a linear combination of the variables in X with integer coefficients.

  ex : $3x+2y-1$.

- A linear formula over X is a boolean combination of inequalities between linear terms over X.

  ex : $3x+2y-1 \geq 0 \wedge y \geq 5$.

- Given a linear formula ψ, we write $[\![\psi]\!]$ for the set of valuations v such that $v \models \psi$.

Effective procedure for Post in RHA

- Linear formulas + quantifiers
  =T($\mathbb{R}$,0,1,+,≤).
  =The theory of reals with addition.

  This theory allows for quantifier elimination.

  ex : "$\forall y \bullet y \geq 5 \rightarrow x+y \geq 7$" is equivalent to "$x \geq 2$".

- A symbolic region of H is a finite set

$$\{ (l,\psi_l) \mid l \in Loc \} \text{ where } [\![\psi_l]\!] \subseteq [\![Inv(l)]\!].$$

Effective procedure for Post in RHA

Given a location $l \in Loc$ and a set of valuations $V \subseteq [X \rightarrow \mathbb{R}]$ such that $V \subseteq Inv(l)$, the forward time closure, noted $\langle V \rangle_l^{\nearrow}$ is the set of valuations that are reachable from some valuation $v \in V$ by letting time pass.

This set is defined as follows:

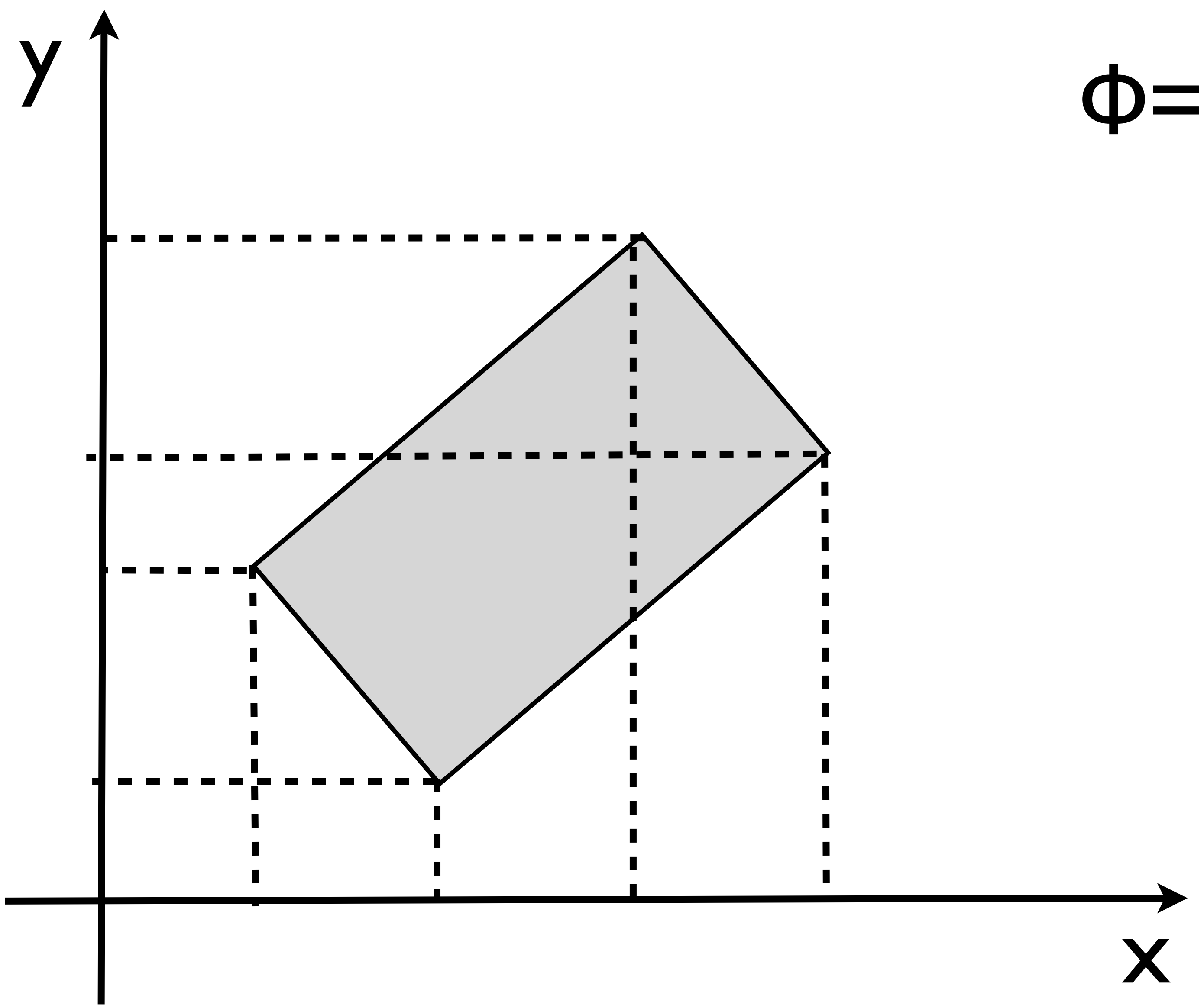
$\langle V \rangle_l^{\nearrow}$ is the set of valuation $v' \in [X \rightarrow \mathbb{R}]$ such that

$$
\begin{aligned}
&\exists \mathbf{v} \in V \bullet \exists t \in \mathbb{R}_{\geq 0} \bullet \forall x \in X \bullet \\
&\qquad \mathbf{v}(x) + t \times \mathbf{Inf}([\![Flow(l)]\!](x)) \leq v'(x) \leq \mathbf{v}(x) + t \times \mathbf{Sup}([\![Flow(l)]\!](x)) \\
&\qquad \wedge v'(x) \in [\![Inv(l)]\!].
\end{aligned}
$$

After **quantifier eliminations**, we get a boolean combination of linear constraints.

An example of time elapsing

Assume $\dot{x}=[1,2]$ and $\dot{y}=1$



$$\Phi=\{ (x,y) \mid \quad x\in[1,4] \wedge y\in[1,6] \wedge y\geq -2x+5 \wedge \ldots \}$$

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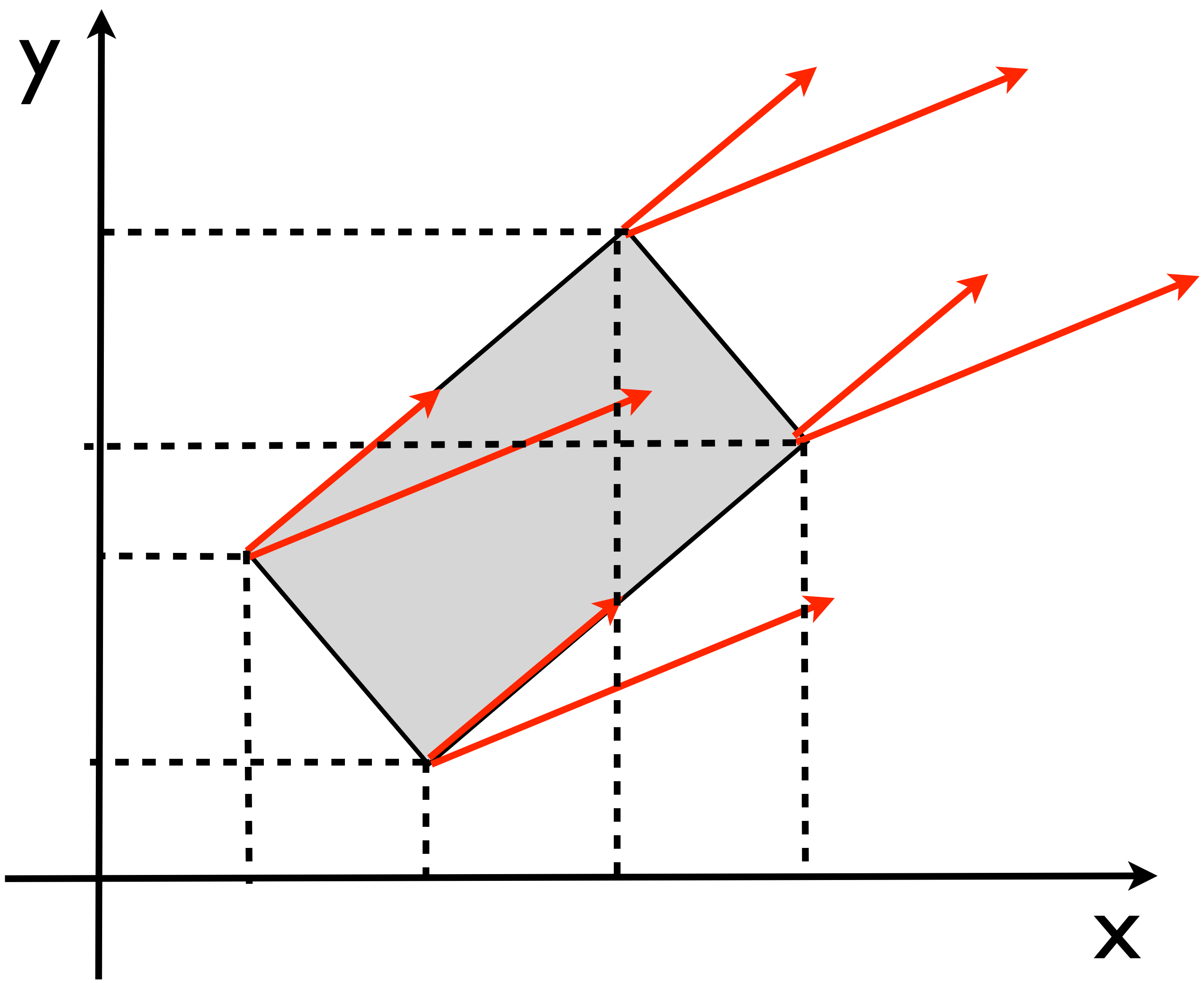

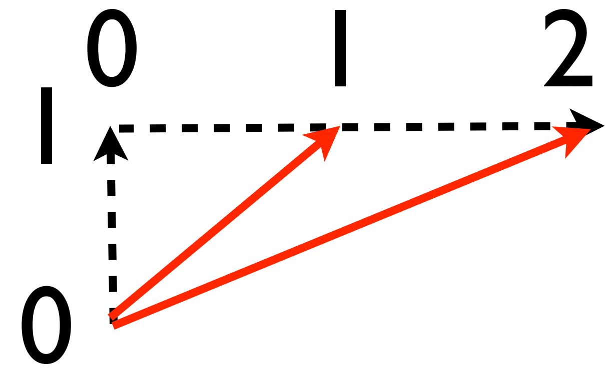

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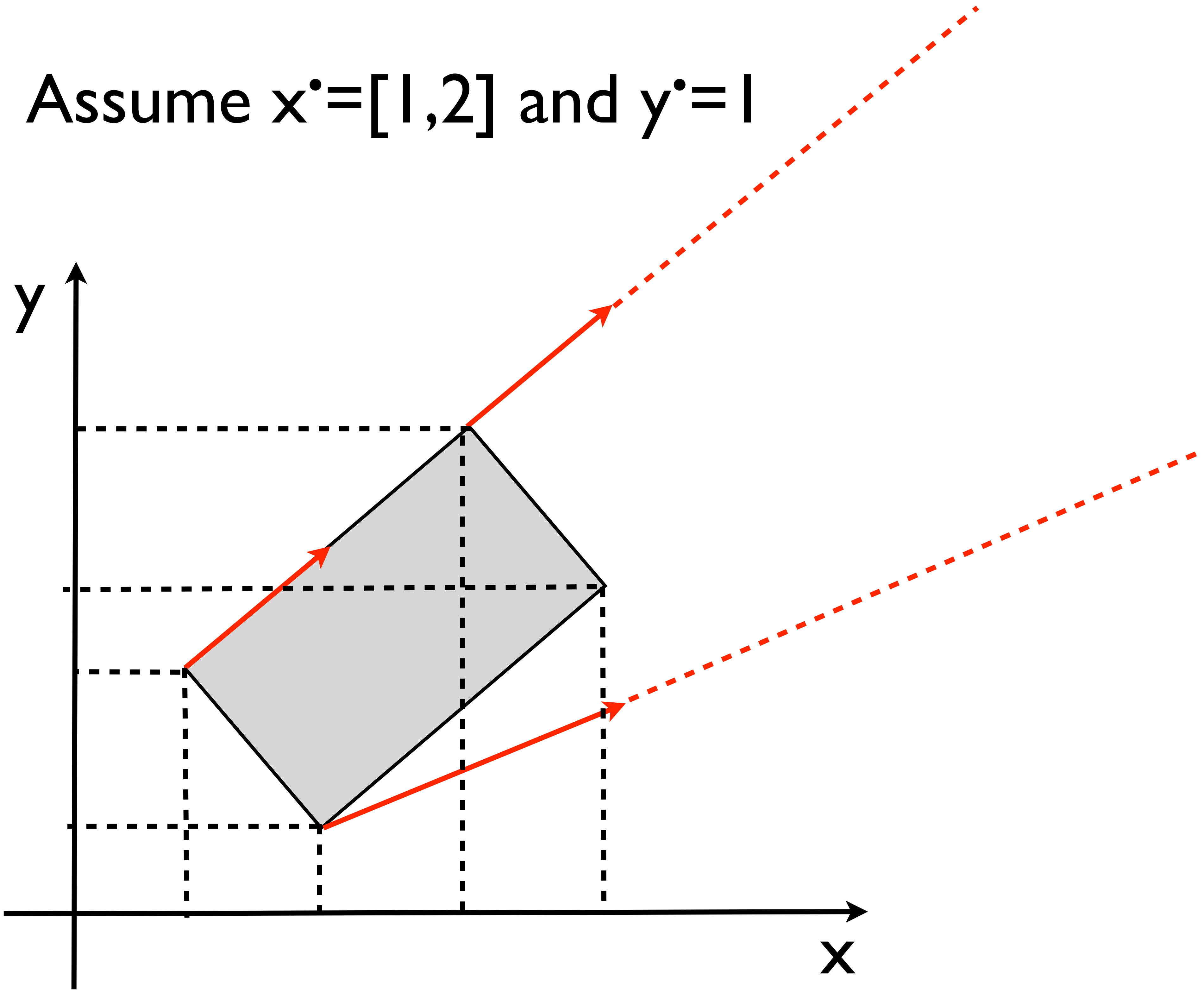

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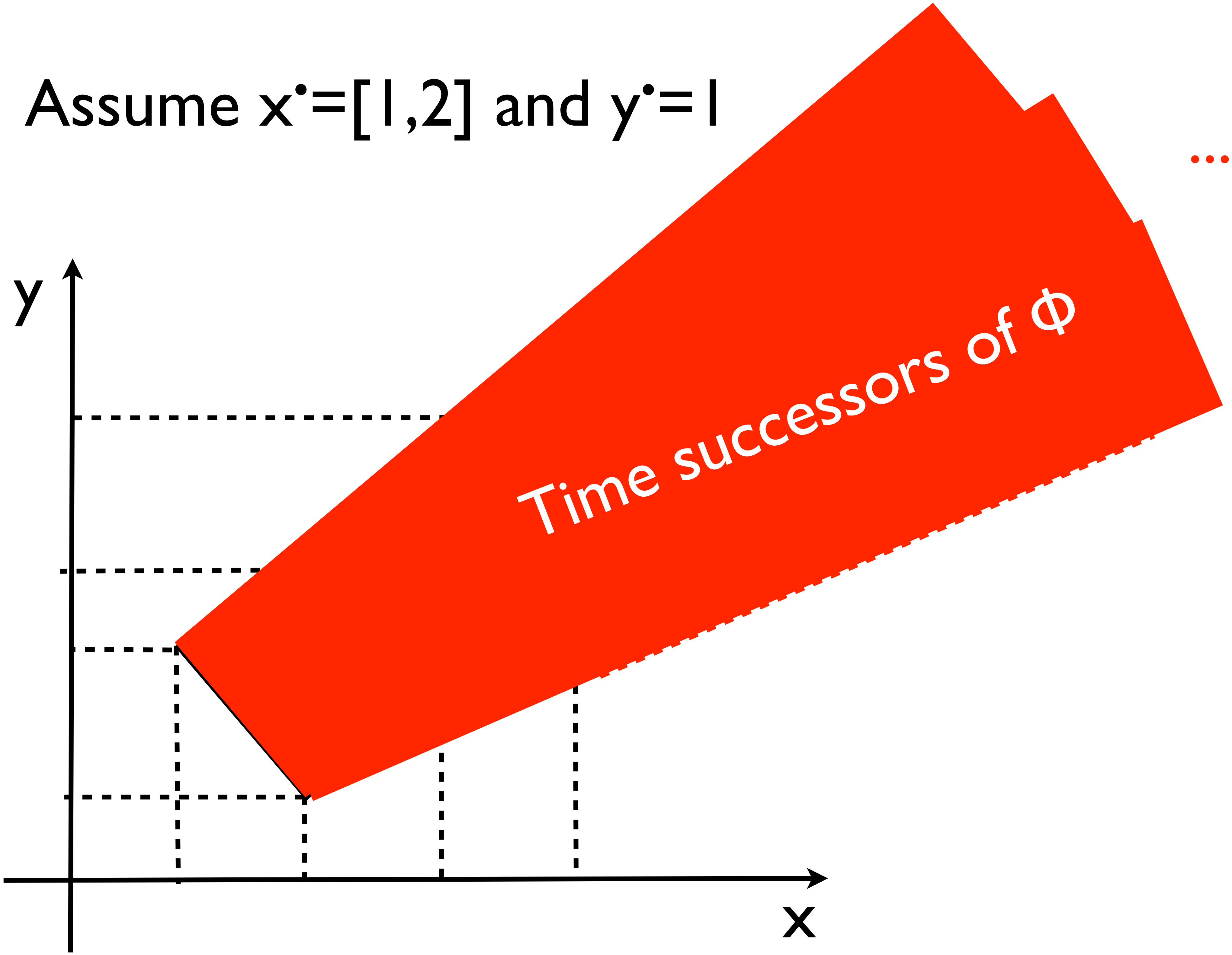

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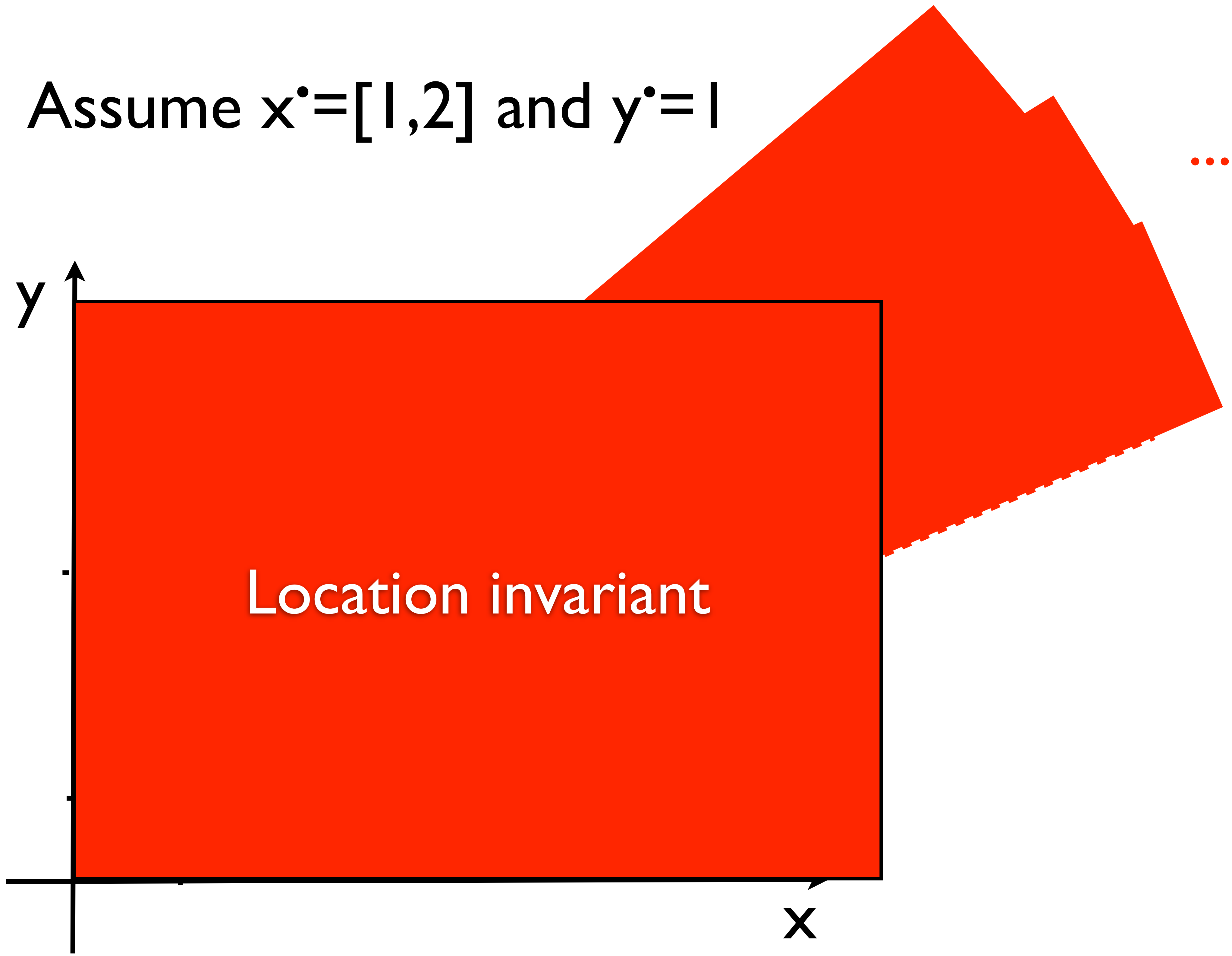

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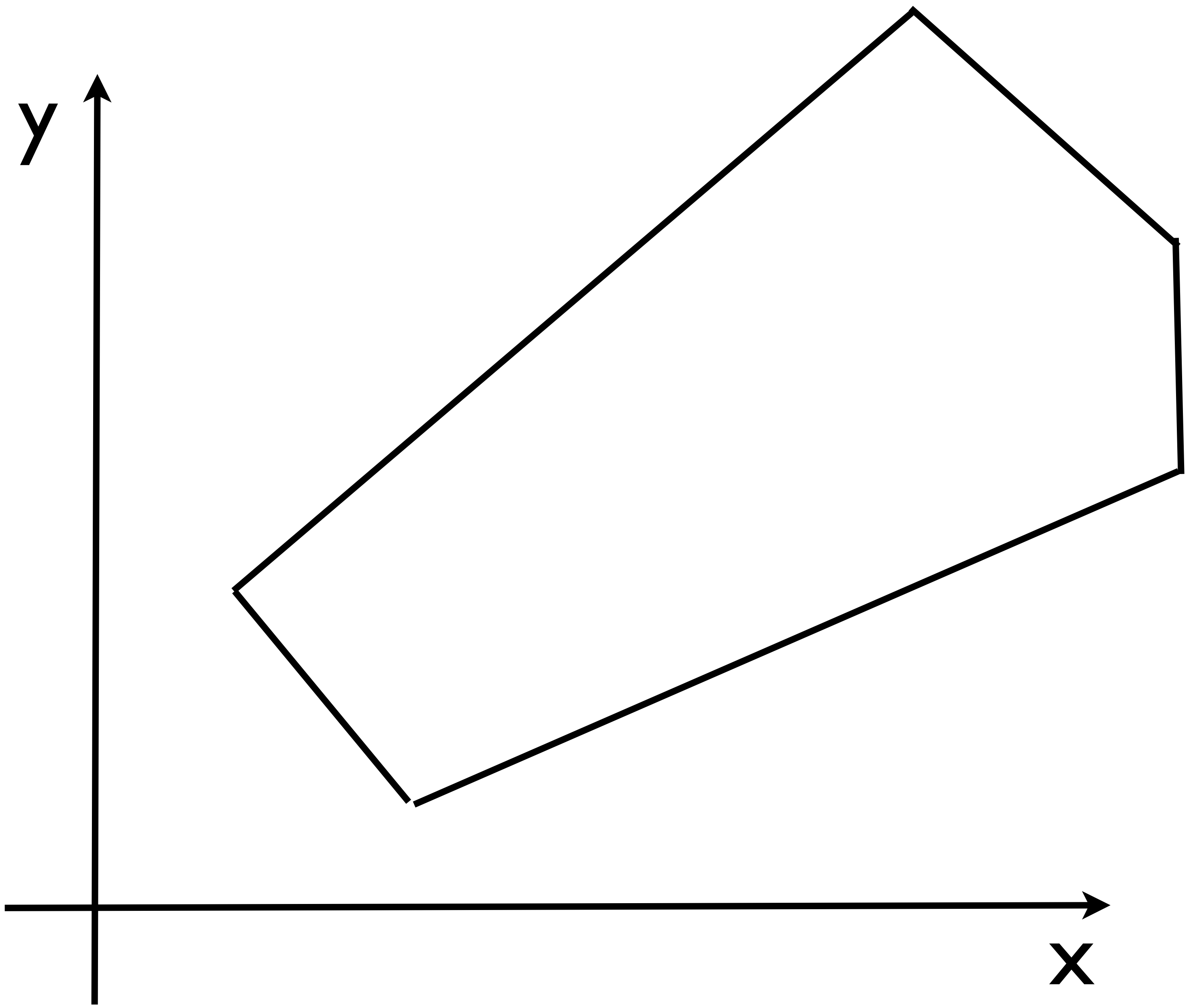

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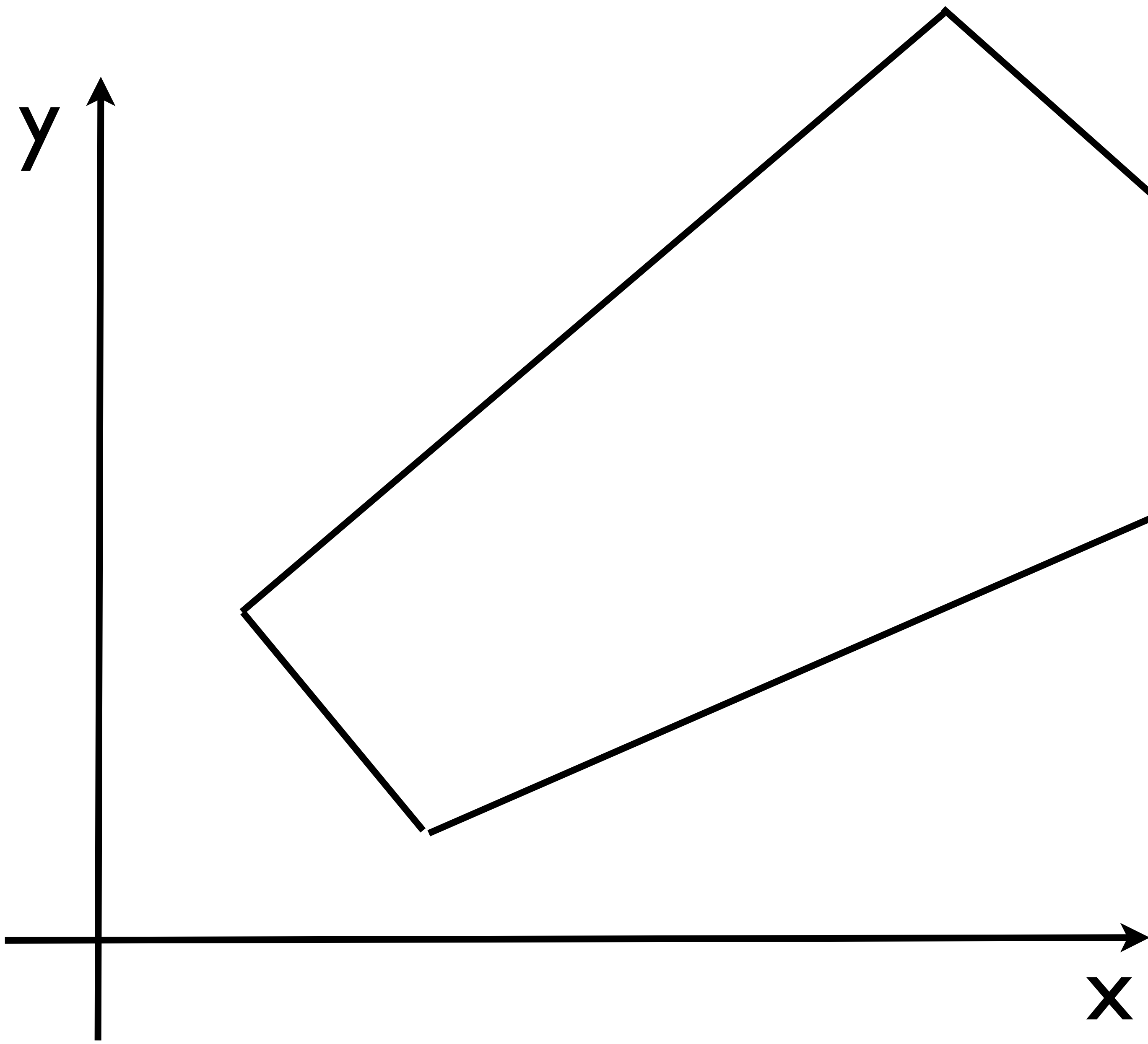
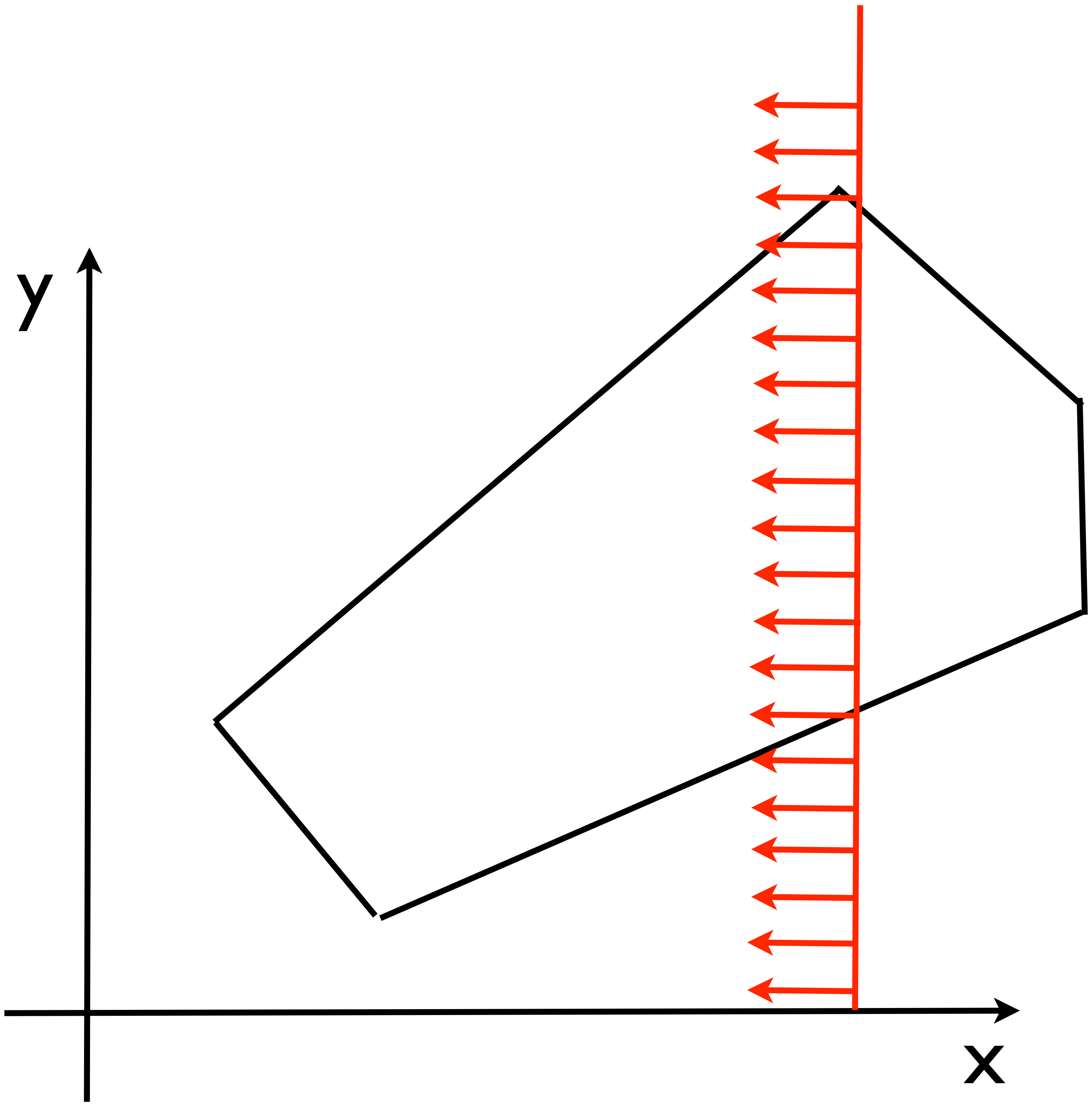


Assume a transition with guard x≤5 and reset of y to zero.

An example of discrete step



Assume a transition with guard $x \leq 5$ and reset of y to zero.

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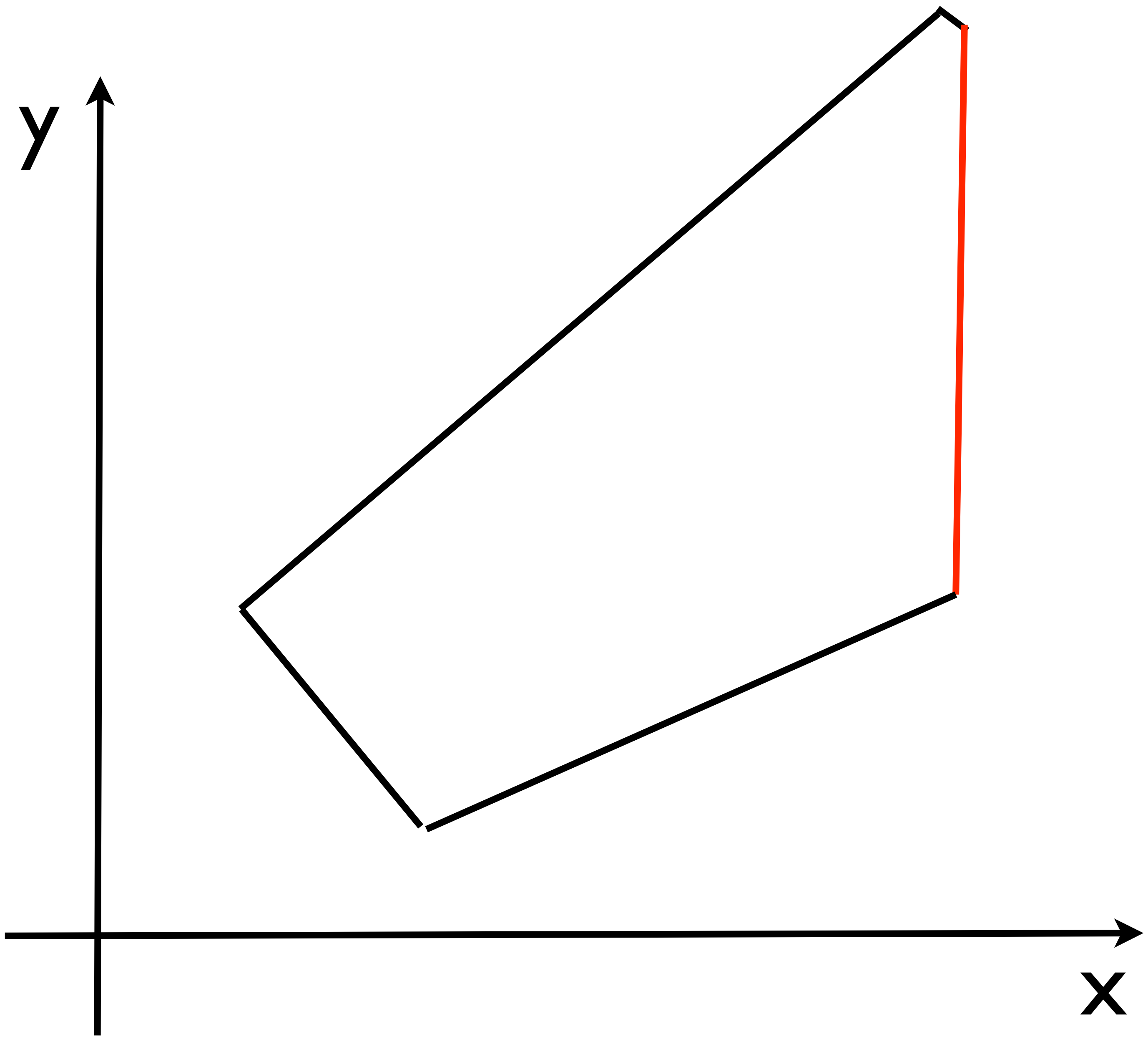

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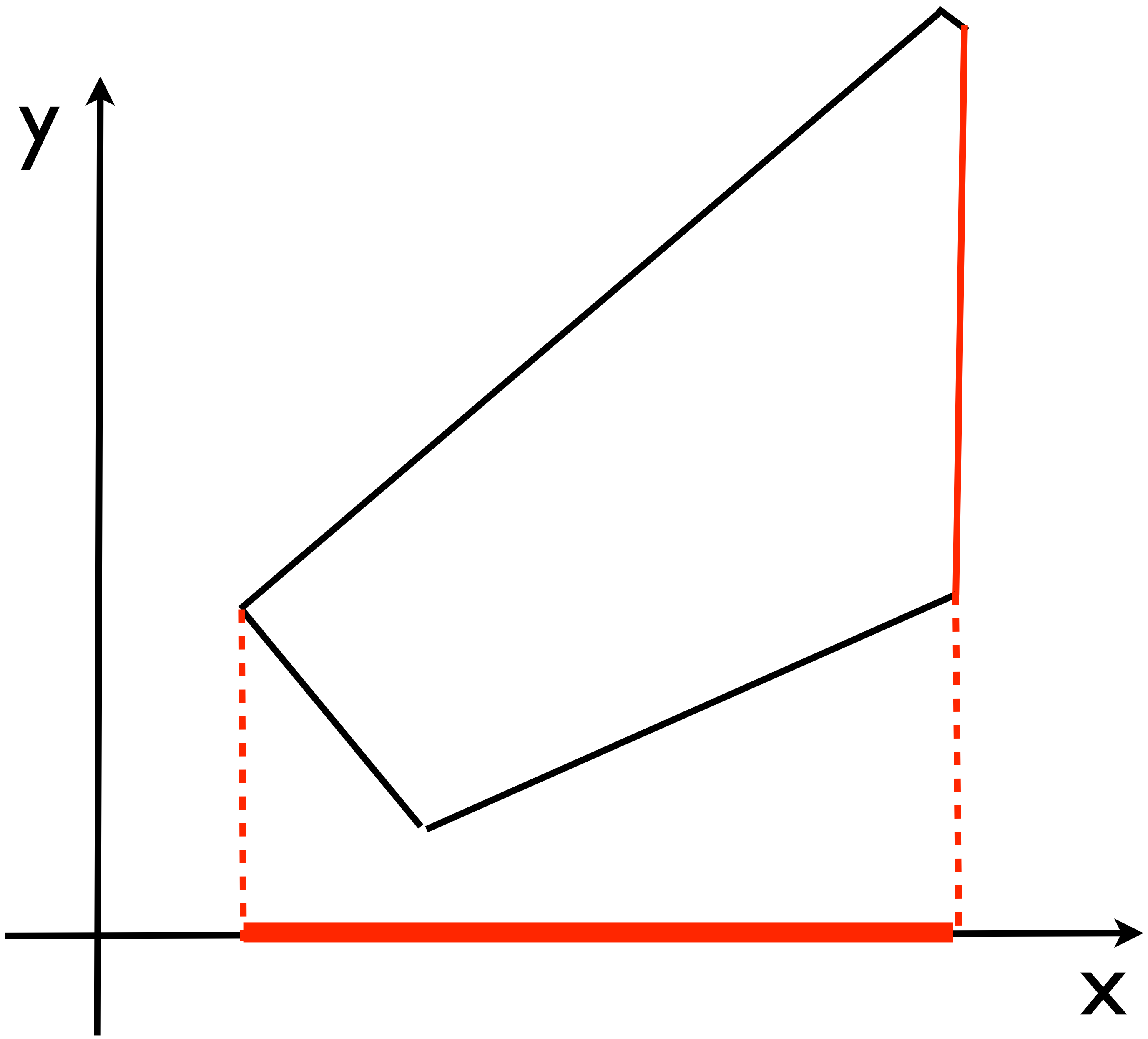

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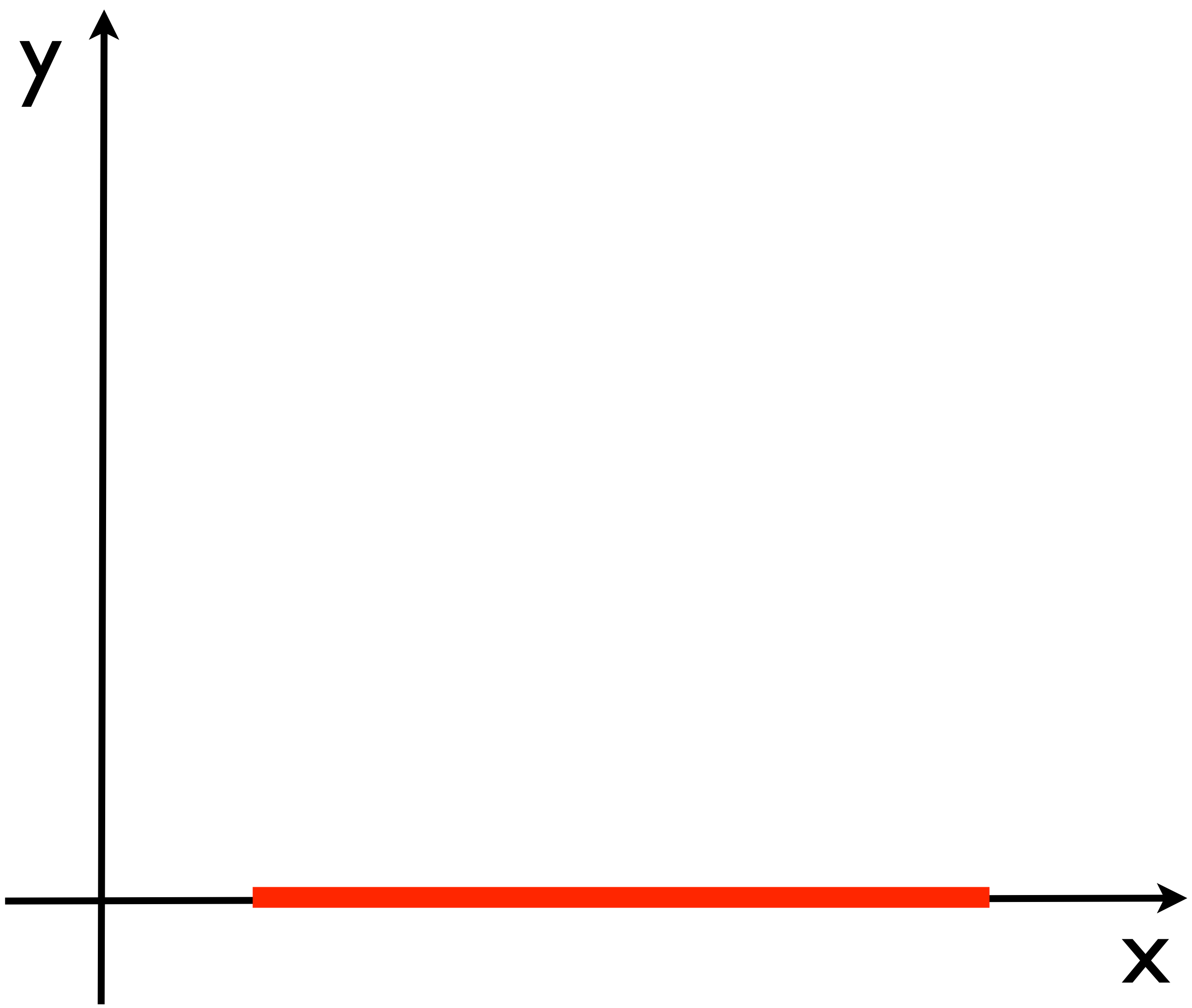

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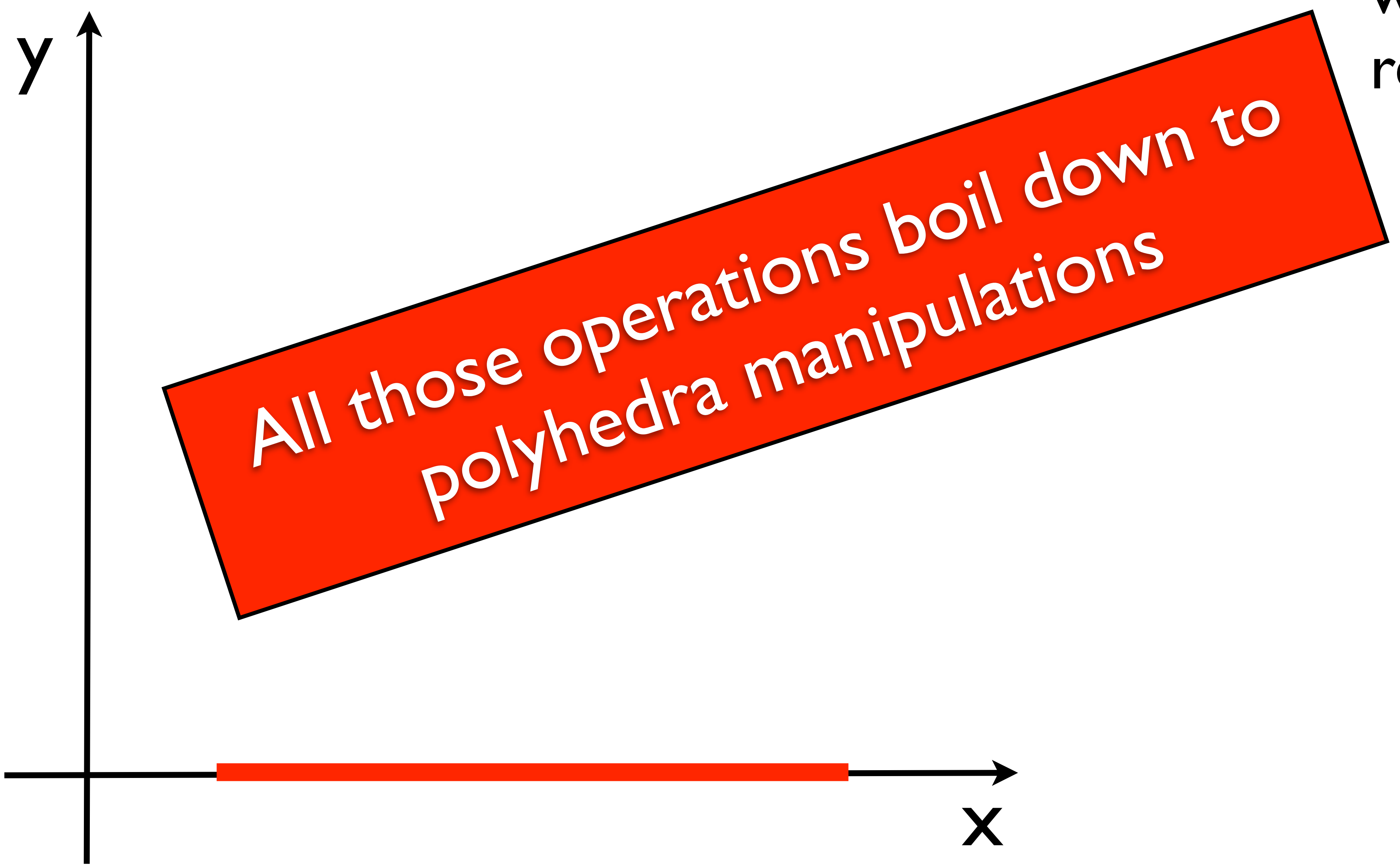

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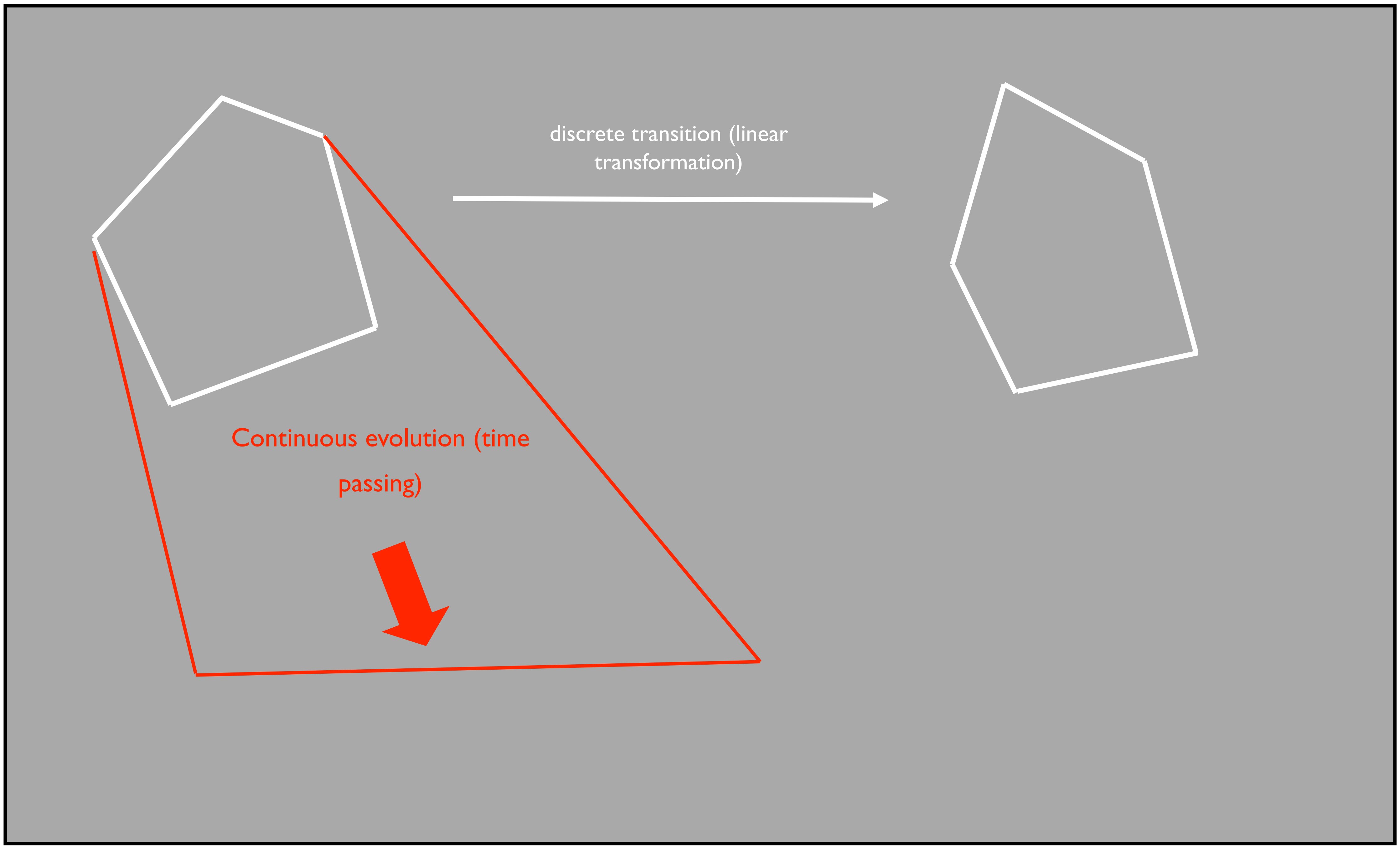
Forward reachability analysis



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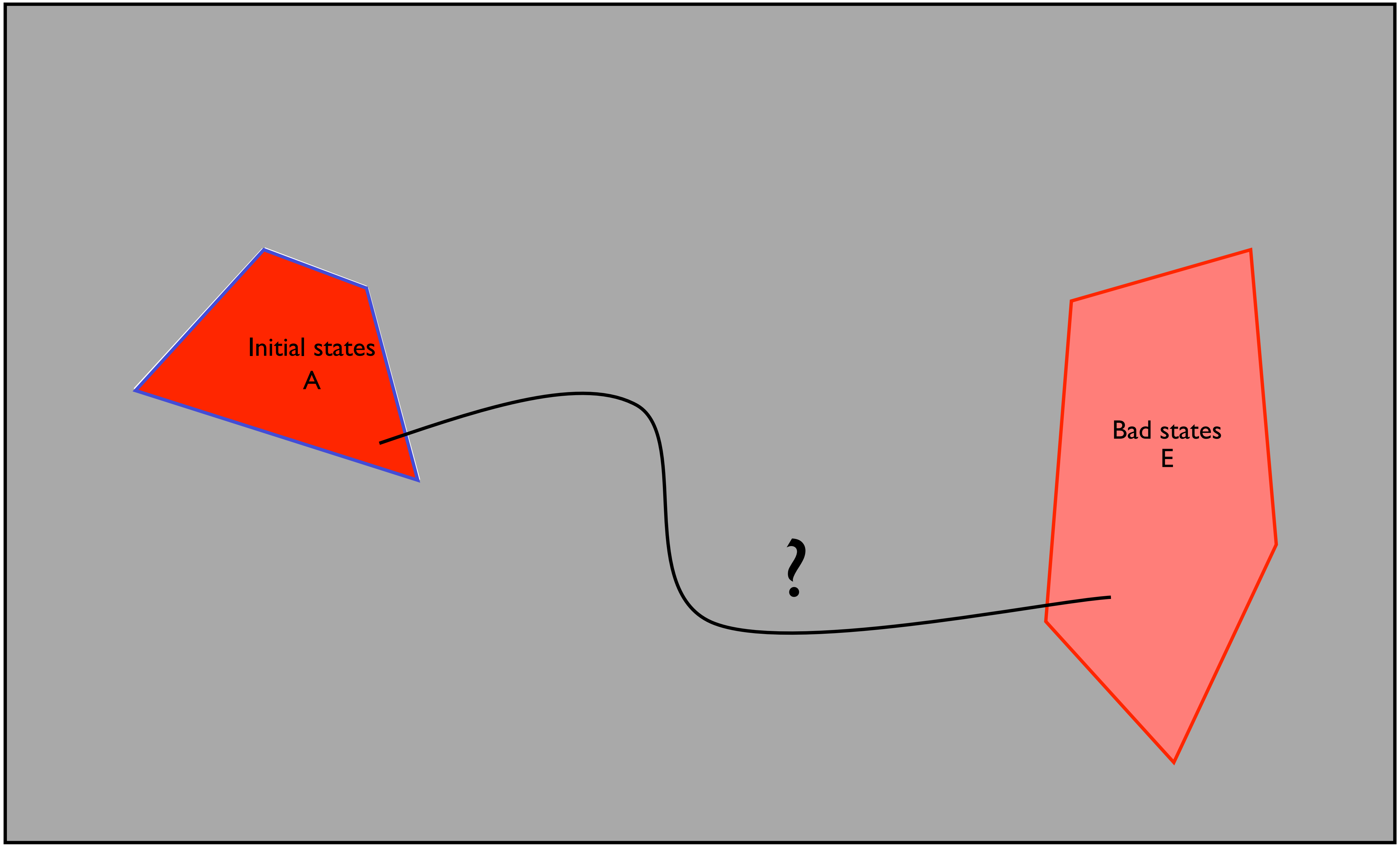

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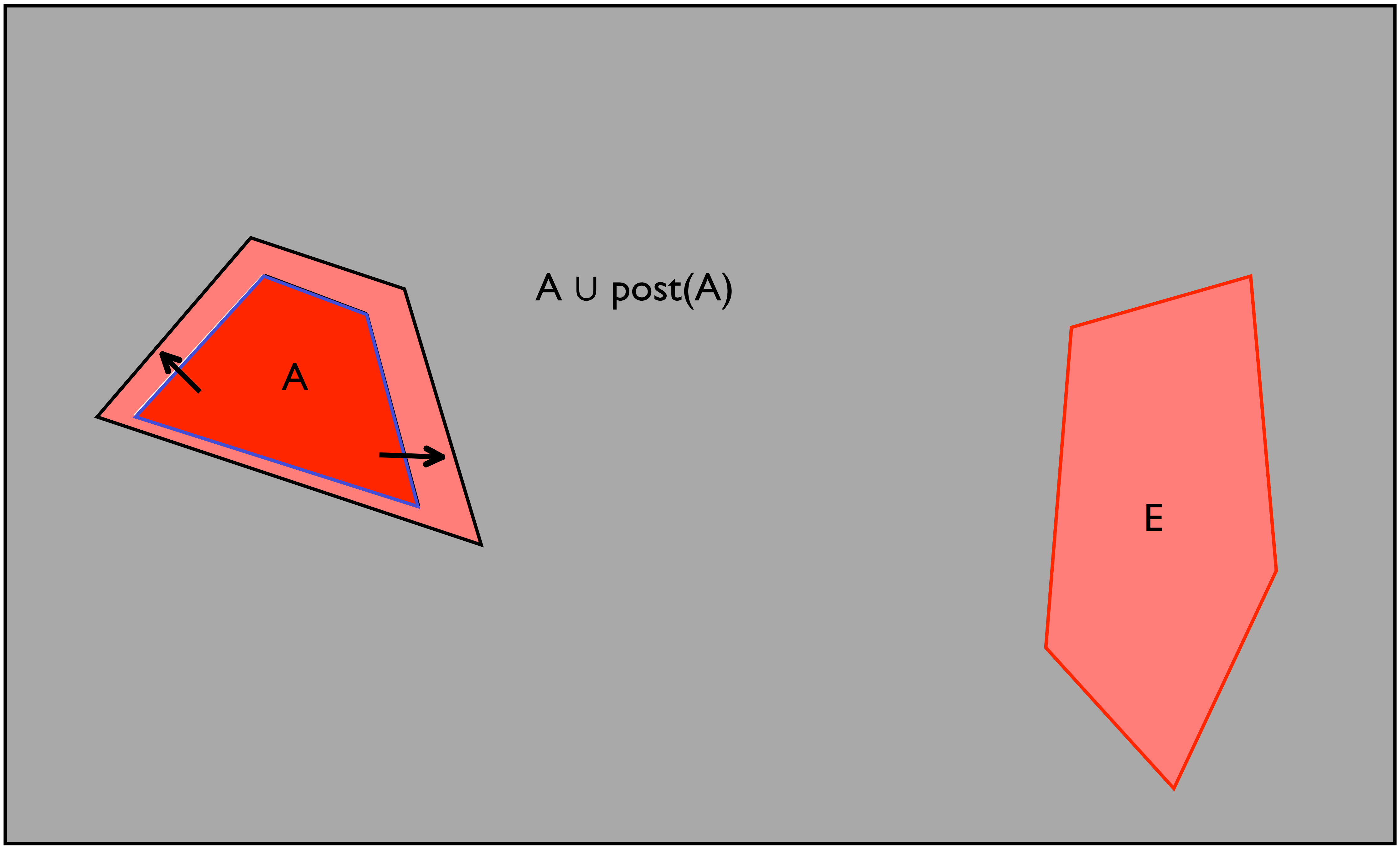

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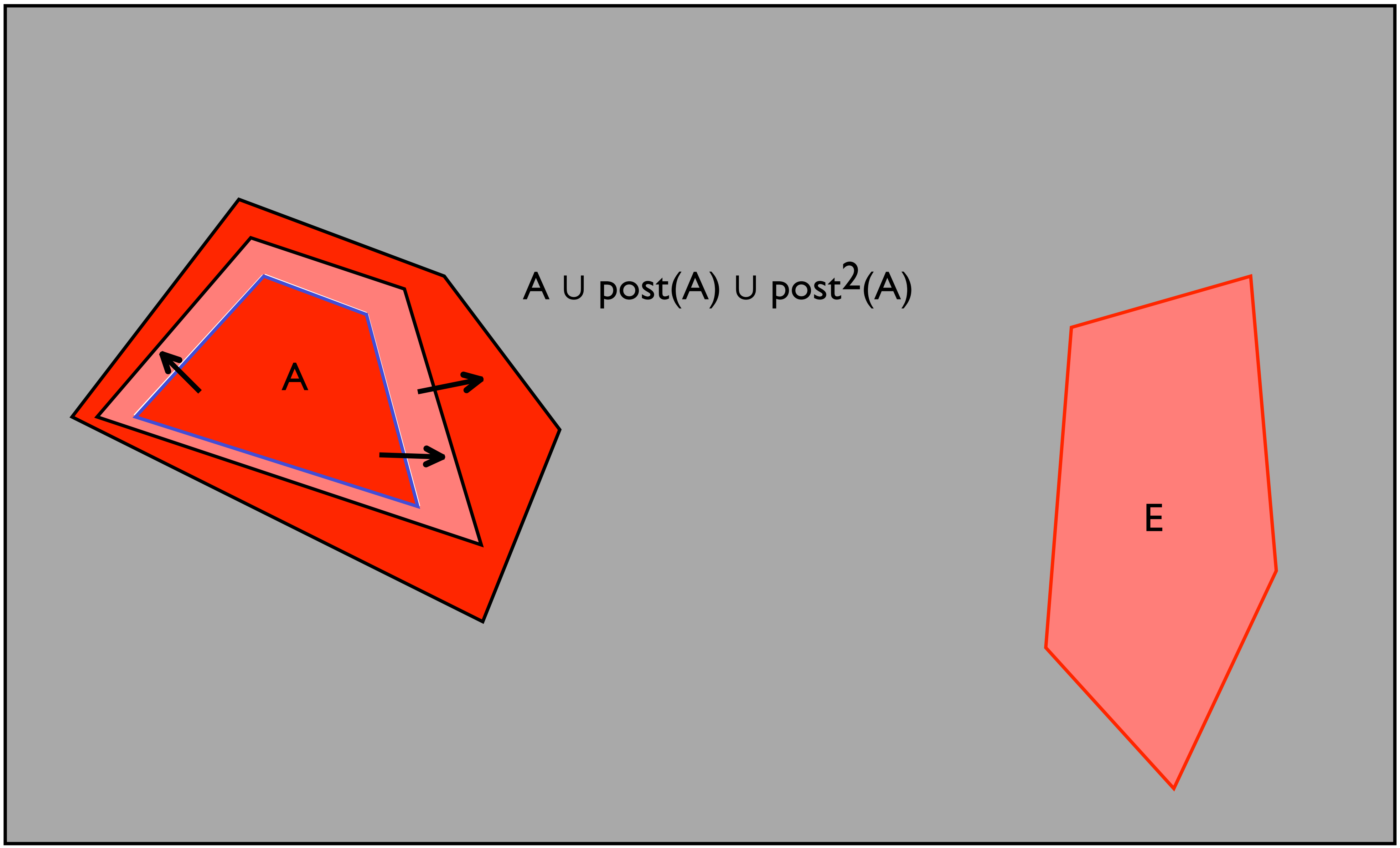

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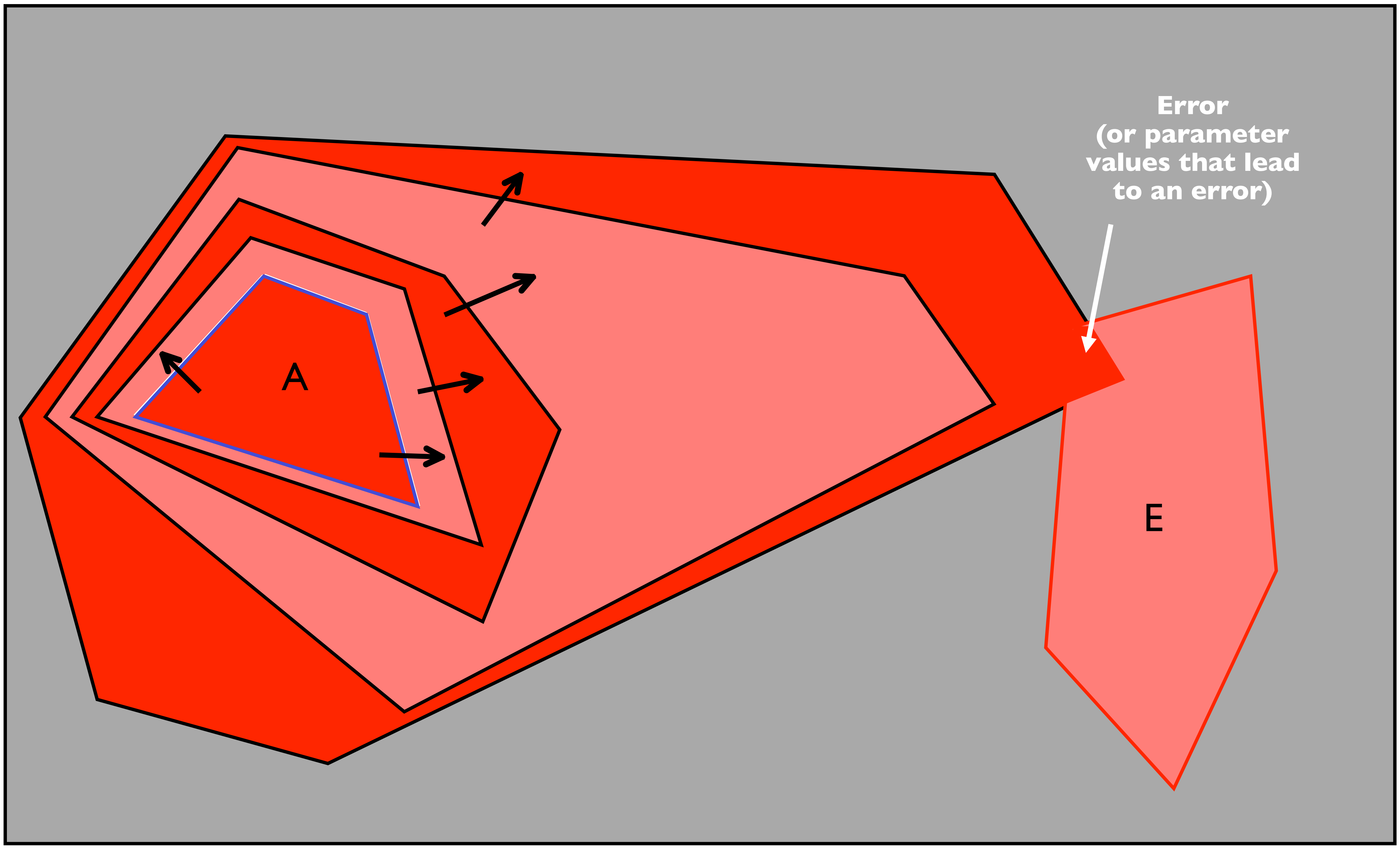

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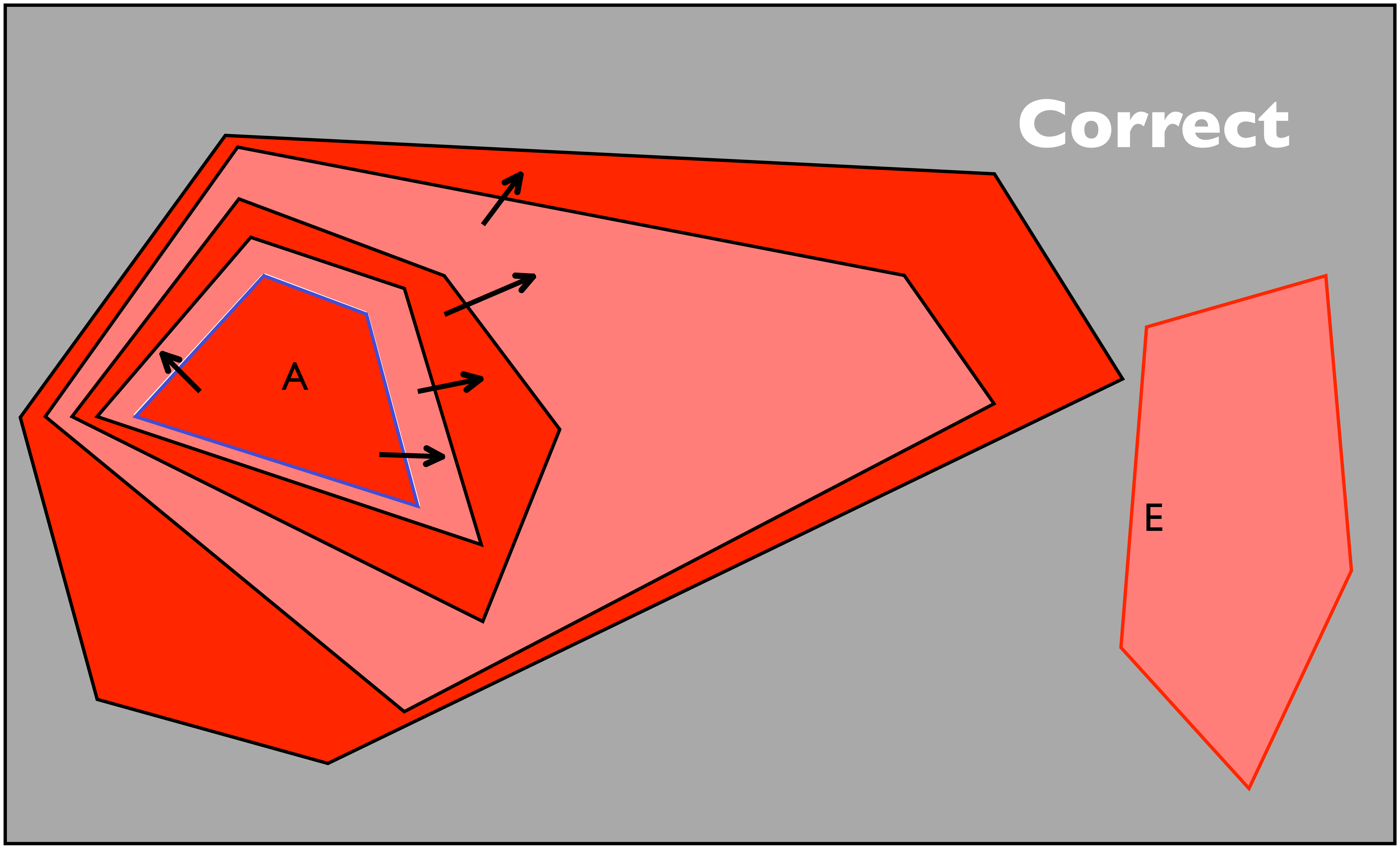

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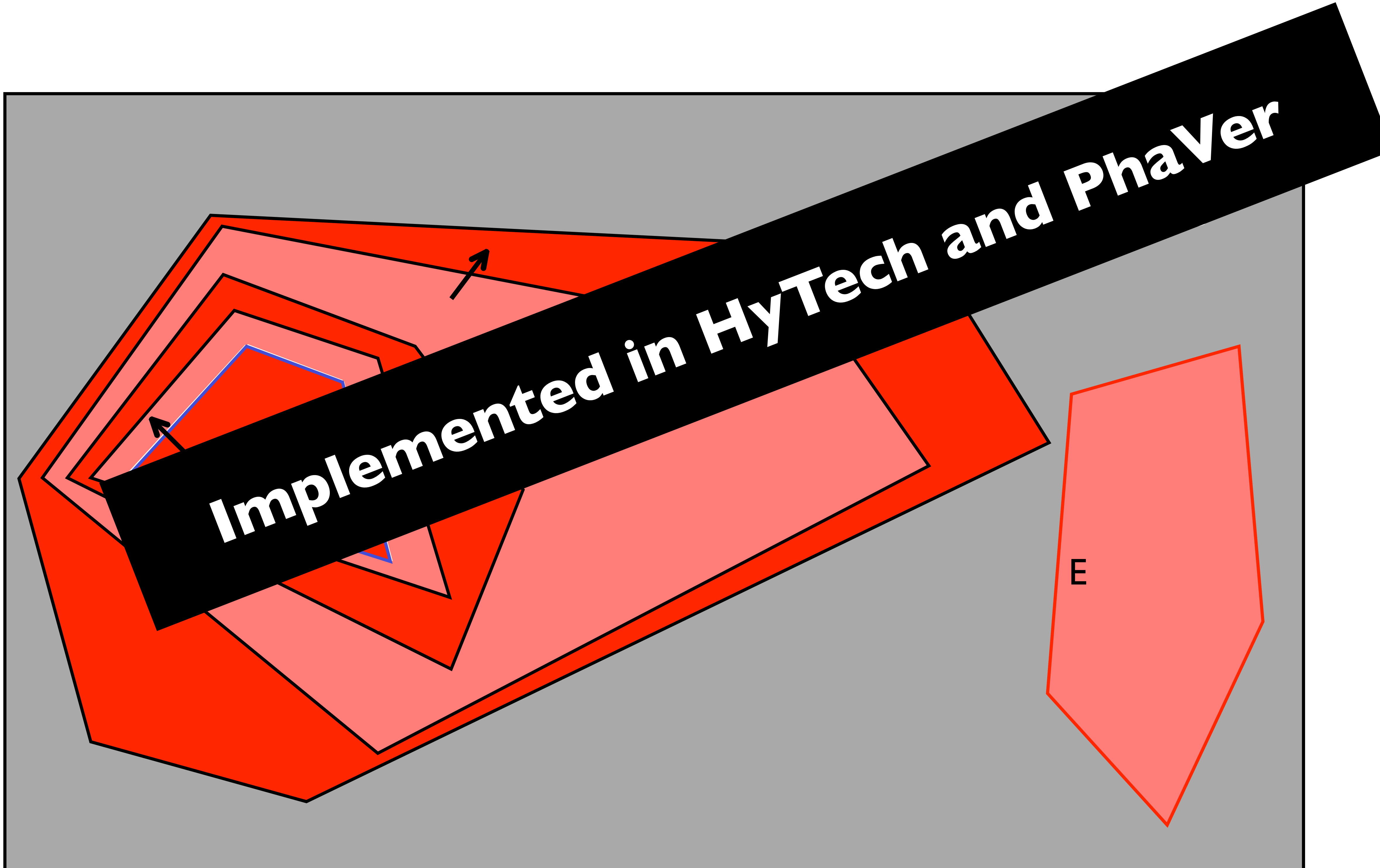


Decidability/ undecidability

Undecidability

Theorem.

The reachability problem for rectangular hybrid automata is undecidable.

This is already the case for **stopwatch automata** ($\dot{x}$=0/1).

Undecidability

Theorem.

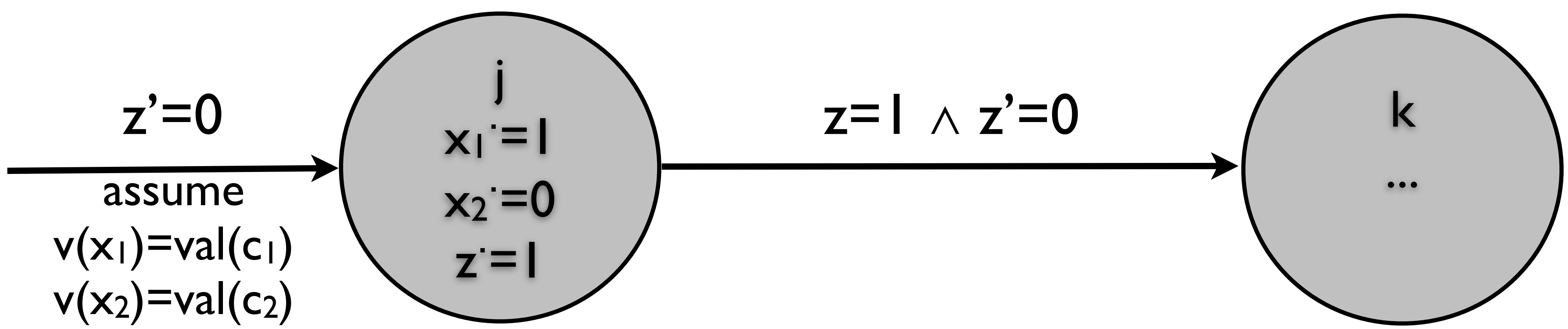
The reachability problem for rectangular hybrid automata is undecidable.

This is already the case for **stopwatch automata** ($\dot{x}$=0/1).

Proof (sketch). By simulation of **two-counter machines** for which the halting problem is undecidable.

To simulate a 2-CM M, we use a RHA with 3 continuous variables.

Let us consider the instruction **j: c_1:=c_1+1; goto k;**



Initialized RHA

- A RHA is **initialized**, if for all discrete jumps (l_1,σ,l_2), and for all variables $x \in X$:
  - either the flow constraints on x in l_1 and l_2 are identical
  - or variable x is updated during the discrete jump from l_1 to l_2

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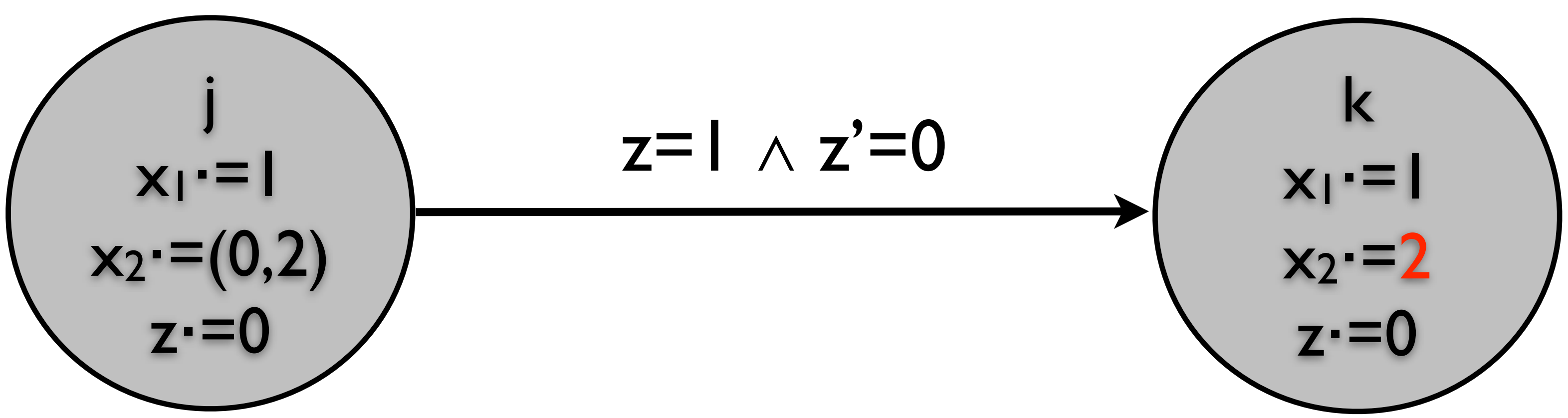

is not initialized

Initialized RHA

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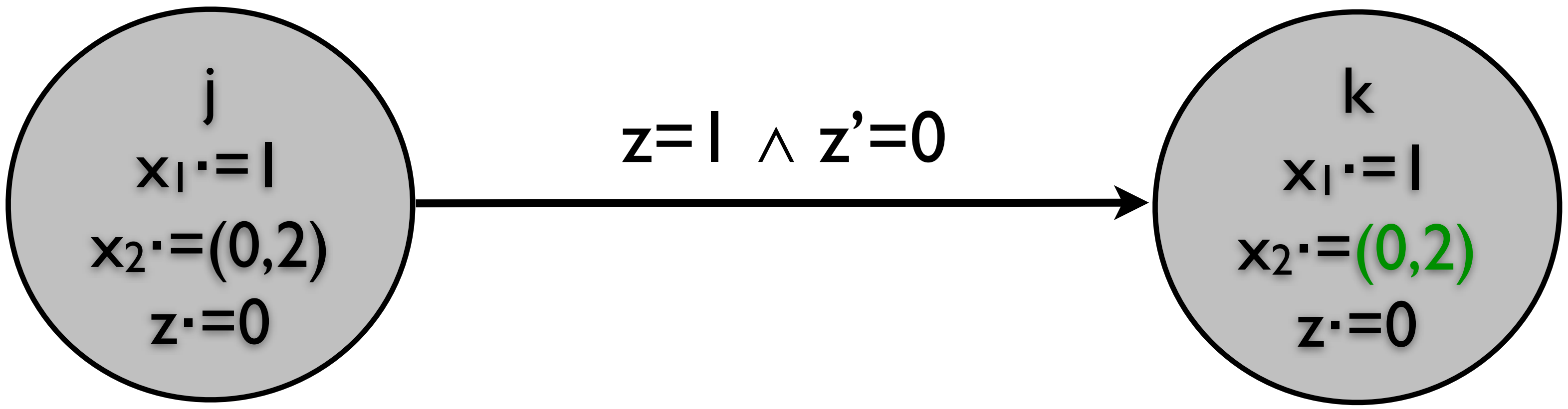

is initialized

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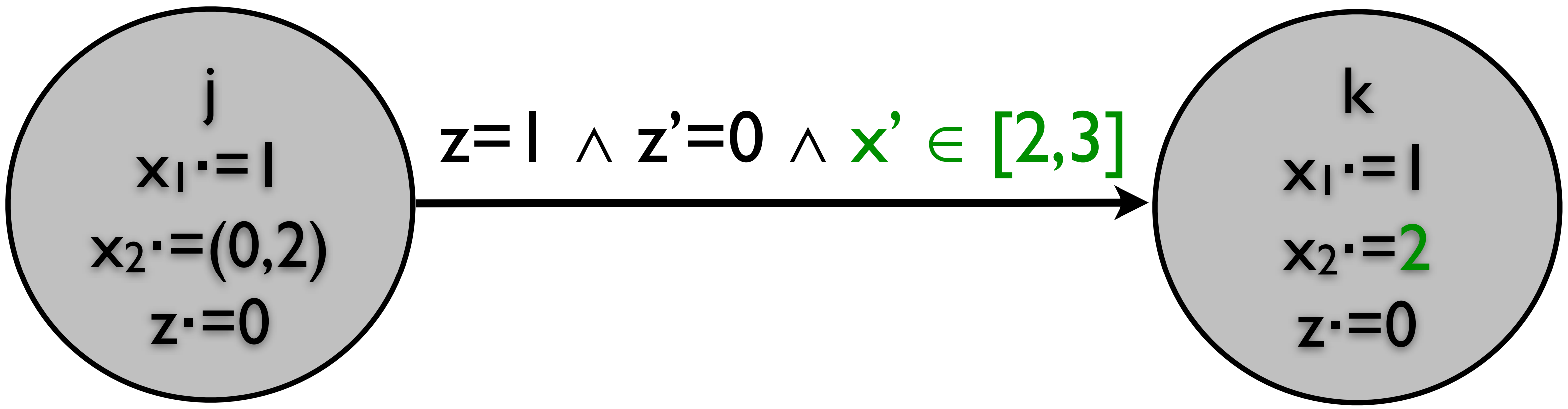

is initialized

Initialized RHA

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  - either the flow constraints on x in l_1 and l_2 are identical
  - or variable x is updated during the discrete jump from l_1 to l_2

  Theorem[HPV96]. The reachability problem (and LTL model-checking problem) is **decidable** for the class of **initialized rectangular automata**.

- Note that Initialized RHA generalizes timed automata

- Existence of finite similarity quotient (init-RHA) and bisimilarity quotient (TA)

Decidability/Undecidability

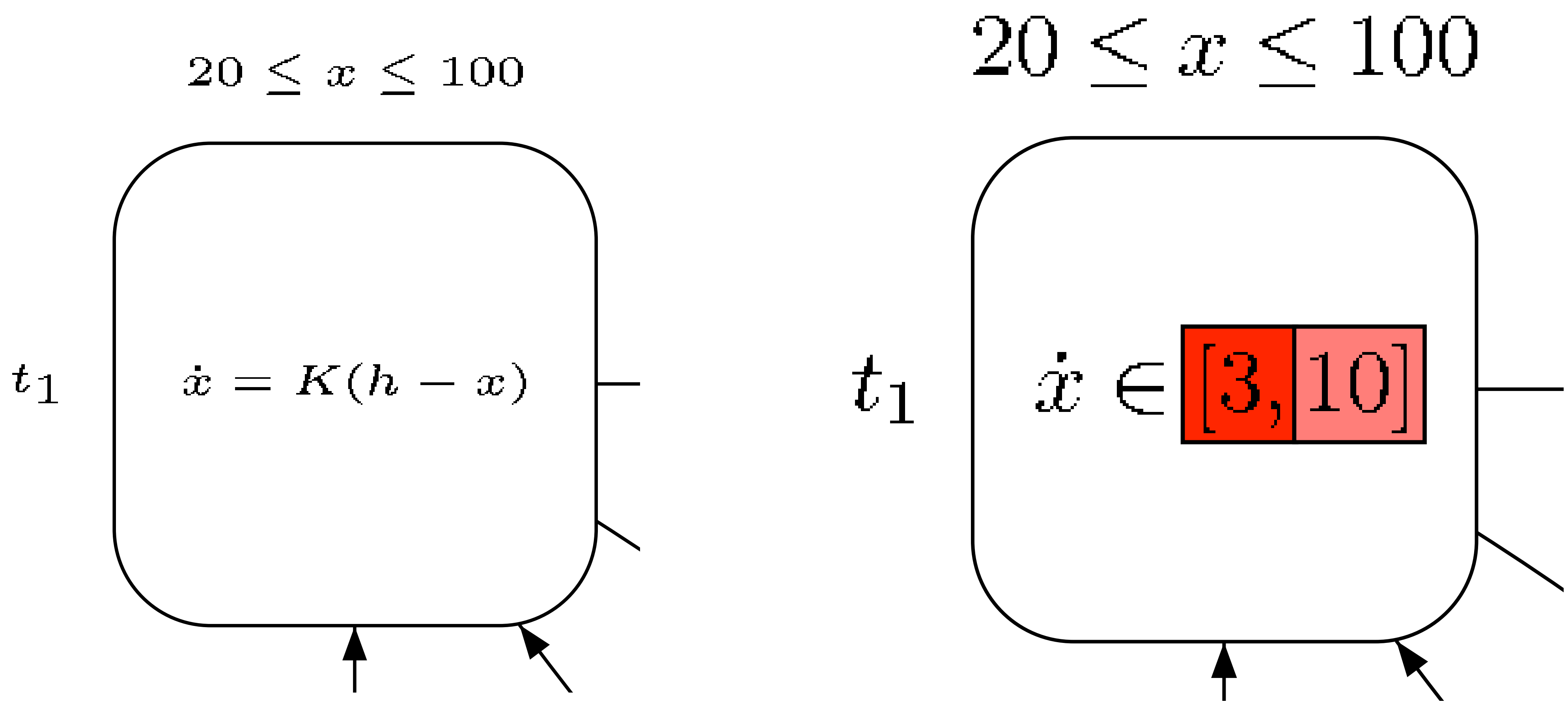
	Reach
Timed automata	☝
Initialized RHA	☝
RHA	☟ **(Stopwatch)**
LHA	☟
Affine HA	☟
O-Minimal HA	☝

Beyong RHA/LHA Approximate Reachability

Rectangular approximations

- **Approximate** complex dynamics with rectangular dynamics
- ... use PhaVer or Hytech for analysis
- Rectangular approximations are often **precise enough**
- For each control mode we **partition** the space into rectangular regions
- Within each region, the flow field is **over-approximated** using rectangular flows
- Those approximations can often be obtained automatically:
  for affine HA $\rightarrow$ solve an **LP** problem
- Approximations can be made **arbitrarily precise** by approximating over suitably small regions of the state space

An example



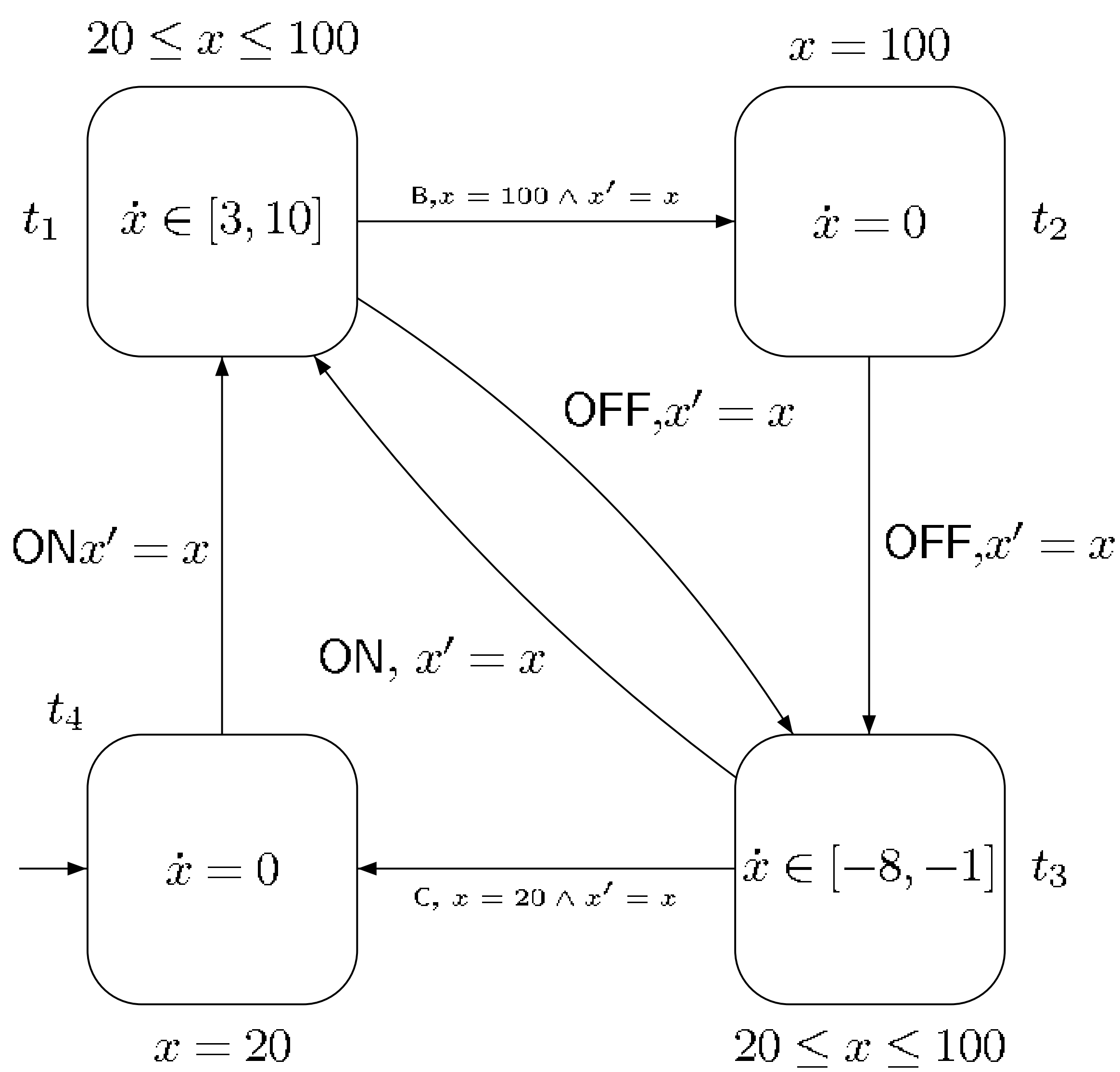
$\text{Max}_{x \in [20,100]}$ K(h-x) = K(h-20) = 0.075(150-20) = 9.75 ≤ 10

$\text{Min}_{x \in [20,100]}$ K(h-x) = K(h-100) = 0.075(150-100) = 3.75 ≥ 3

These are LPs

An example

- Applying this computation for each location, we get the following **rectangular approximation** of the tank:



Over-approximations and correctness

- Let us note **RectOver**(H) the rectangular over-approximation obtained using the previous method;
- RectOver(H) is a **over-approximaiton** of the original system in the following sense:

$$\mathbf{Path_F}(\llbracket \mathbf{H} \rrbracket) \subseteq \mathbf{Path_F}(\llbracket \mathbf{RectOver(H)} \rrbracket)$$

- **Transfert of correctness** from overapproximations:

if $\mathbf{Path_F}(\llbracket \mathbf{RectOver(H)} \rrbracket) \cap \mathbf{BadPaths} = \emptyset$
then $\mathbf{Path_F}(\llbracket \mathbf{H} \rrbracket) \cap \mathbf{BadPaths} = \emptyset$

Over-approximations and correctness

- When over-approximating the behavior of a system, we face the possibility to get **false negatives** during verification;
- Indeed, the set of behaviors of the **over-approximation** is a **superset** of the behaviors of the original system...
- ...so if we have that

$$\mathbf{Path_F([\![RectOver(H)]\!]) \cap BadPaths \neq \emptyset}$$

it is **not** nessarily the case that

$$\mathbf{Path_F([\![H]\!]) \cap BadPaths \neq \emptyset}$$

Candidate counter examples

- A path $\lambda = s_0 \tau_0 s_1 \tau_1 \ldots \tau_{n-1} s_n$ is an **candidate counter example** if
  - $\lambda \in [\![\text{OverRect}(H)]\!] \cap \text{BadPaths}$
- When facing a candidate counter example, we check if the counter example is realizable in the original model, so we ask:
  - $\lambda \in^{?} [\![H]\!]$

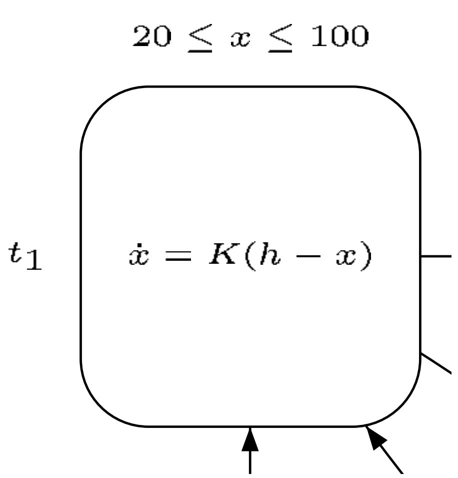
    This test is possible for larger class than rectangular automata, i.e. affine/polynomial hybrid automata.
- If $\lambda \in [\![H]\!]$, then we have found a **real** counter example i.e., the a Bad path in the original HA H.

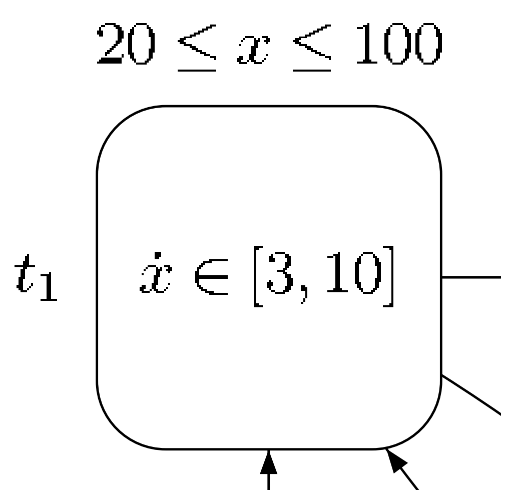
Spurious counter-examples

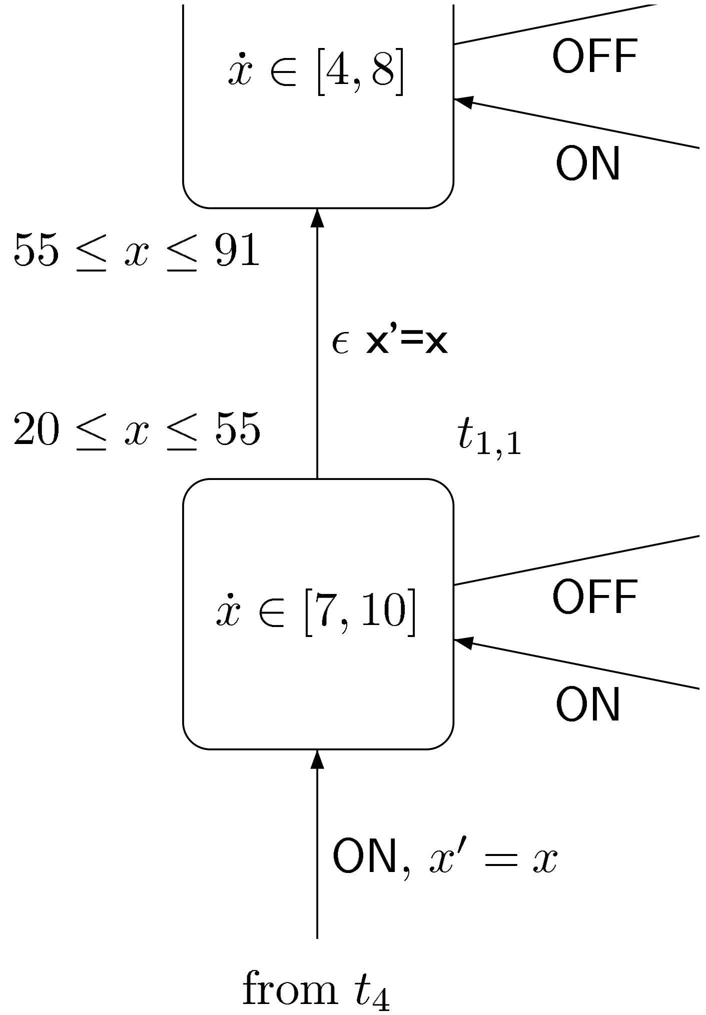
- If $\lambda \notin [\![H]\!]$, then λ is a **spurious counter example** i.e.:
  - $\lambda \in [\![\text{OverRect}(H)]\!] \cap \text{BadPaths}$
  - $\lambda \notin [\![H]\!]$
- In this case, we must **refine** OverRect(H) in order to eliminate the counter example.
- There is a large research effort in the CAV community on the so-called **counter-example based abstraction refinement**, and variants.

Abstraction refinement

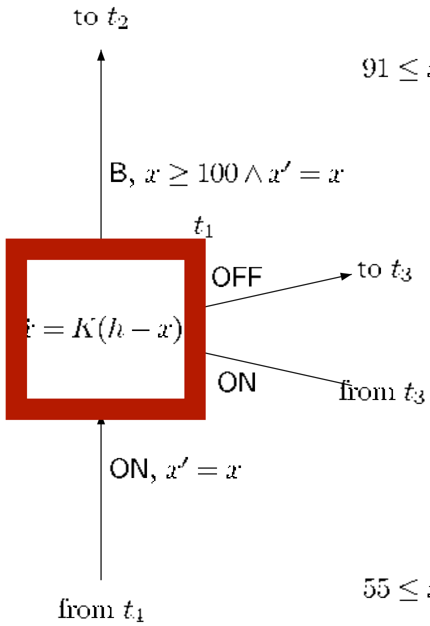
- In presence of **spurious counter examples**, we **refine** the rectangular approximation by **splitting** locations to decorate them with smaller rectangular regions.



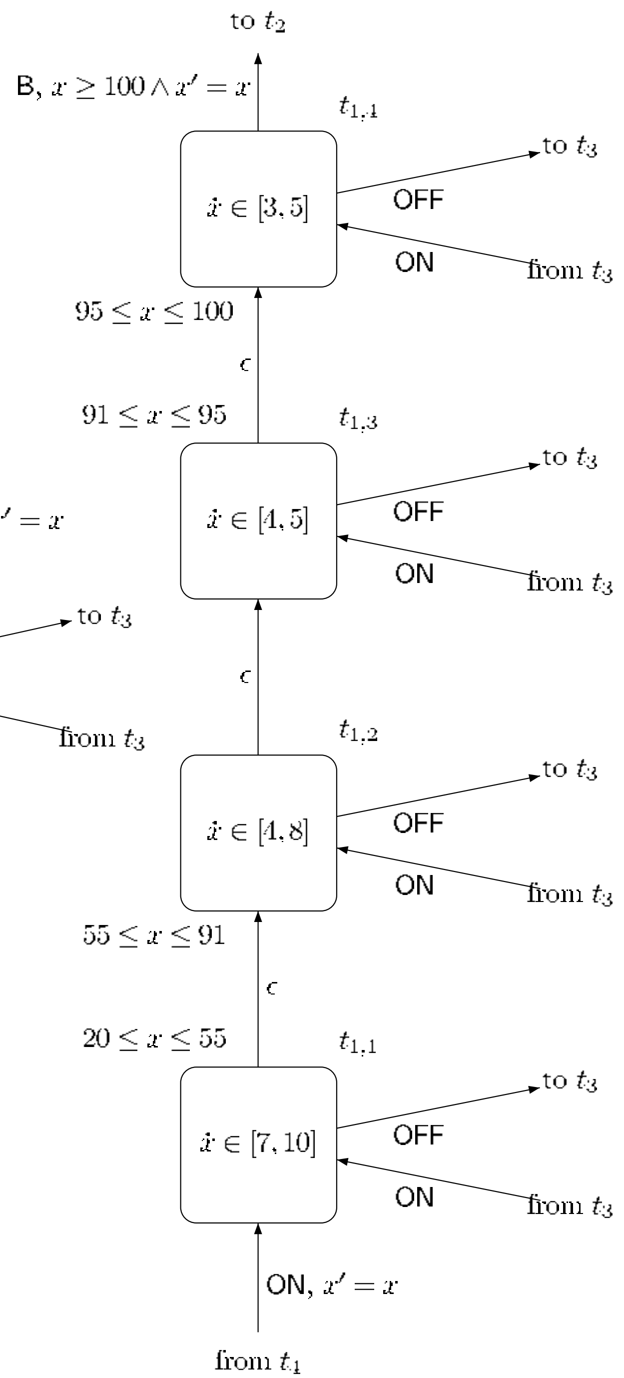




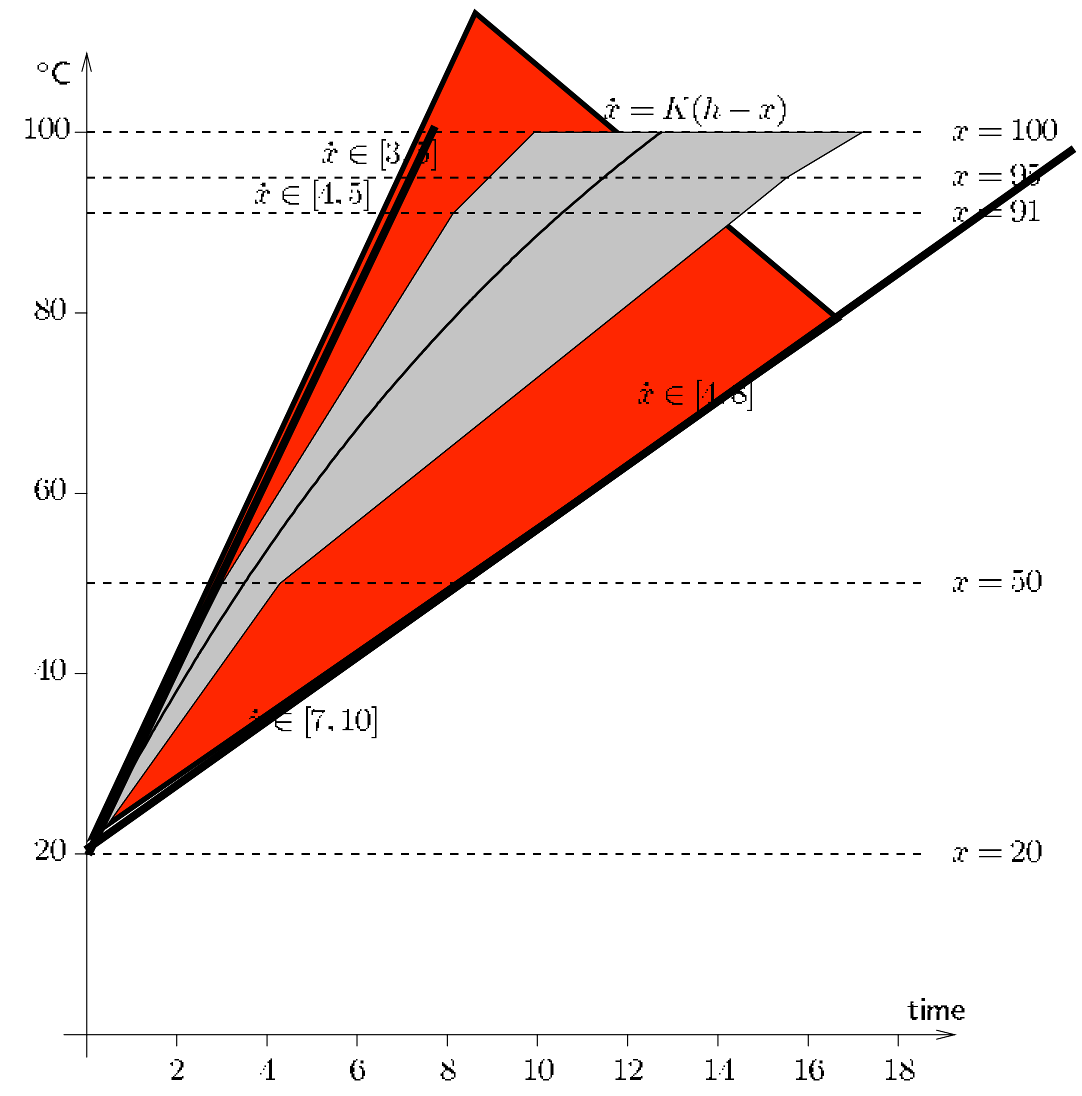
Example



to t_2
B, $x \geq 100 \wedge x' = x$
t_1
OFF
to t_3
$\dot{x} = K(h-x)$
ON
from t_3
ON, $x' = x$
from t_1



to t_2
B, $x \geq 100 \wedge x' = x$
$t_{1,4}$
$\dot{x} \in [3,5]$
to t_3
OFF
ON
from t_3
$95 \leq x \leq 100$
c
$91 \leq x \leq 95$
$t_{1,3}$
$\dot{x} \in [4,5]$
$t_{1,2}$
$\dot{x} \in [4,8]$
$55 \leq x \leq 91$
$20 \leq x \leq 55$
$t_{1,1}$
$\dot{x} \in [7,10]$
ON, $x' = x$
from t_1



°C
100
80
60
40
20
$\dot{x} = K(h-x)$
$\dot{x} \in [3,5]$
$\dot{x} \in [4,5]$
$\dot{x} \in [4,8]$
$\dot{x} \in [7,10]$
$x = 100$
$x = 95$
$x = 91$
$x = 50$
$x = 20$
time
2
4
6
8
10
12
14
16
18

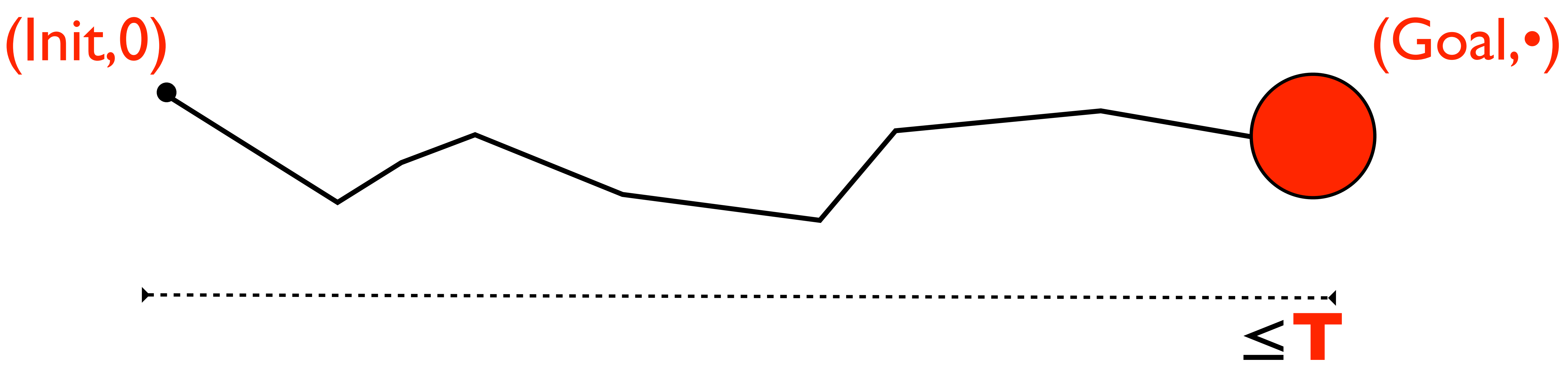
Time-bounded Reachability

Time Bounded Reachability

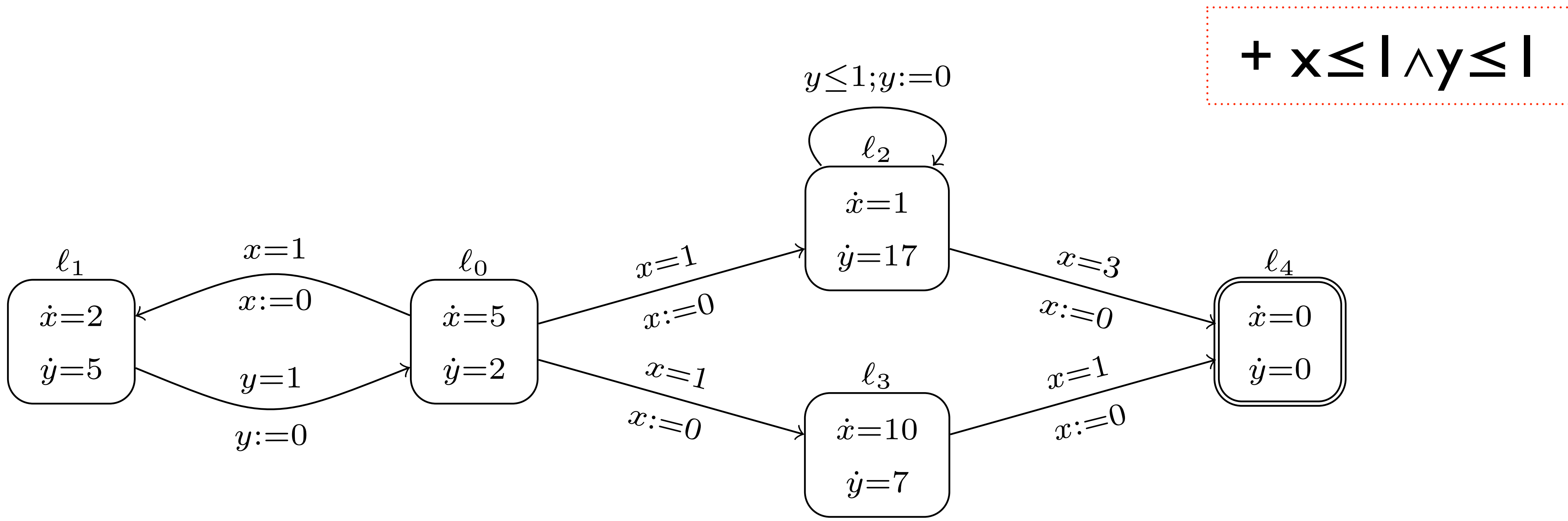
Definition

- Given an LHA H=(X,Loc,Edges,Rates,Inv,Init)
  - a location Goal∈Loc and
  - a time bound **T**∈ℕ

The **time bounded reachability problem** is to decide if ∃ρ=(Init,0)→(Goal,•) of H with duration(ρ)≤**T**.



Time Bounded Reachability



- This automaton is **non-initialized**, but

  (I) **non-negative** rates

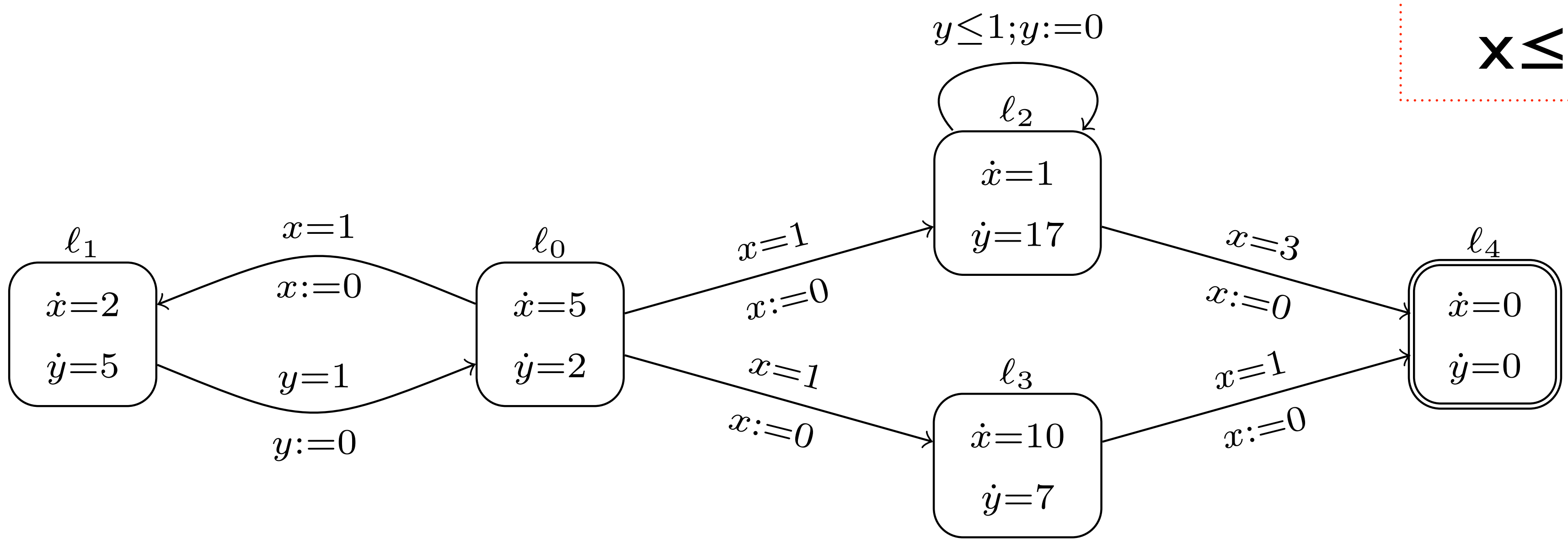
  (II) **diagonal free**

- class **RHA**⊕ for which we show **decidability** of TBR

Time Bounded Reachability

+ Invariant

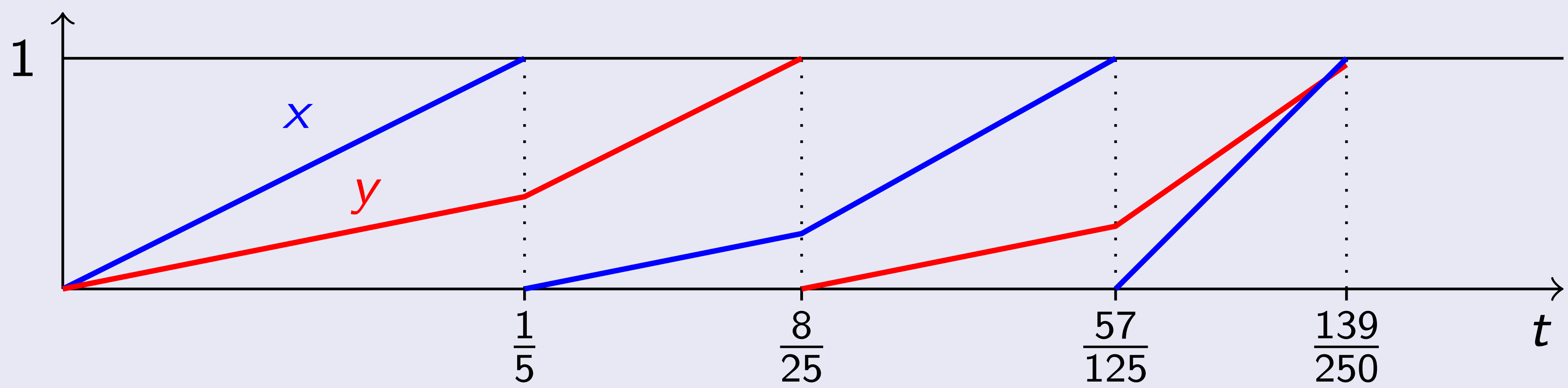
$x \leq 1 \wedge y \leq 1$



Goal

$T \leq 1$

$$(\ell_0,0,0) \xrightarrow{\frac{1}{5}, e_{01}} \left(\ell_1,0,\tfrac{2}{5}\right) \xrightarrow{\frac{3}{25}, e_{10}} \left(\ell_0,\tfrac{6}{25},0\right) \xrightarrow{\frac{17}{125}, e_{03}} \left(\ell_3,0,\tfrac{34}{125}\right) \xrightarrow{\frac{1}{10}, e_{34}} \left(\ell_4,0,\tfrac{243}{250}\right).$$



Additional hypothesis (wlog)

- RHA⊕:
  - **non-negative** rates
  - **diagonal free**

- All variables are bounded by 1
  - (L,2.1,4.7) is encoded by ((L,2,4),0.1,0.7)
  - **Only** guards of the form **x<1**, **x=1**
  - As soon as a clock reaches value 1, it is reset

Bounding the number of transitions

Our goal:

- Given ρ an execution of H reaching Goal from (L_0,x_0) within **T** time units.
- We want to build an execution ρ' of H such that :

  -ρ' reaches Goal from (L_0,x_0) within **T** time units
  -the number of transitions of ρ' is **bounded** by a constant depending only of H and **T**

Solution:

① Simple observation: bounding the number of equalities

② Bounded witness between equalities

Bounding number of equalities

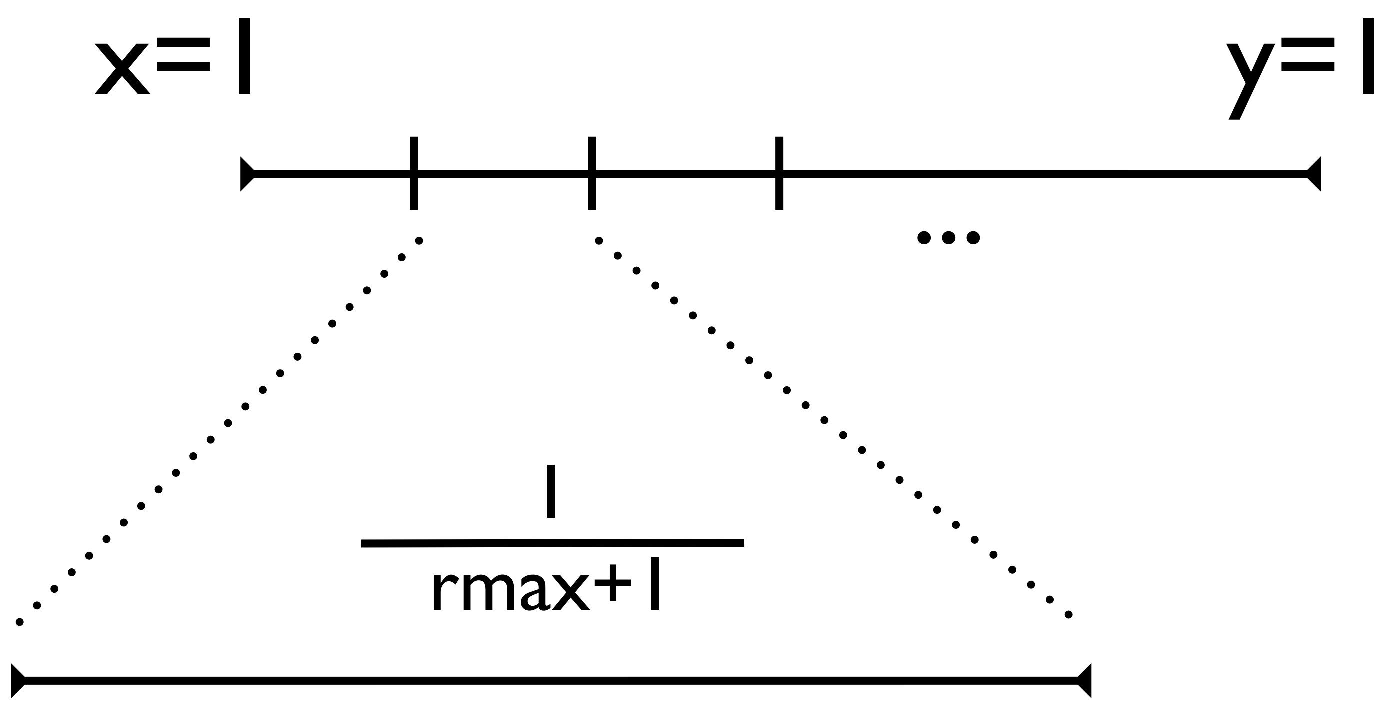
Proposition

- Let H be an RHA⊕ with a set of variables X
- Let ρ be a **T**-time bounded run of H
- Then ρ contains at most |X|•rmax•**T** transitions guarded by an equality

Proof:

- Use bounded time hypothesis
- False for transitions not guarded by an equality

Bounding between two equalities



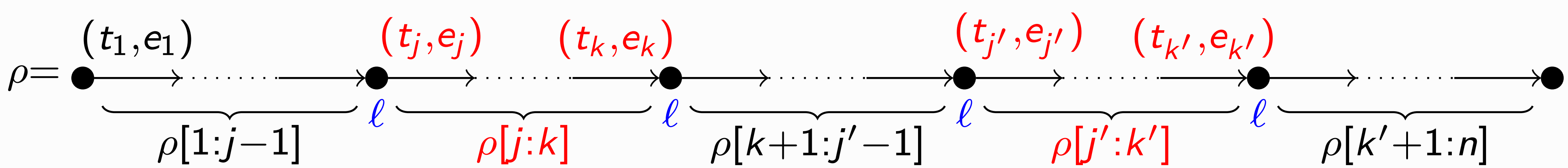
-no equality
-bounded time

→ shorten witness

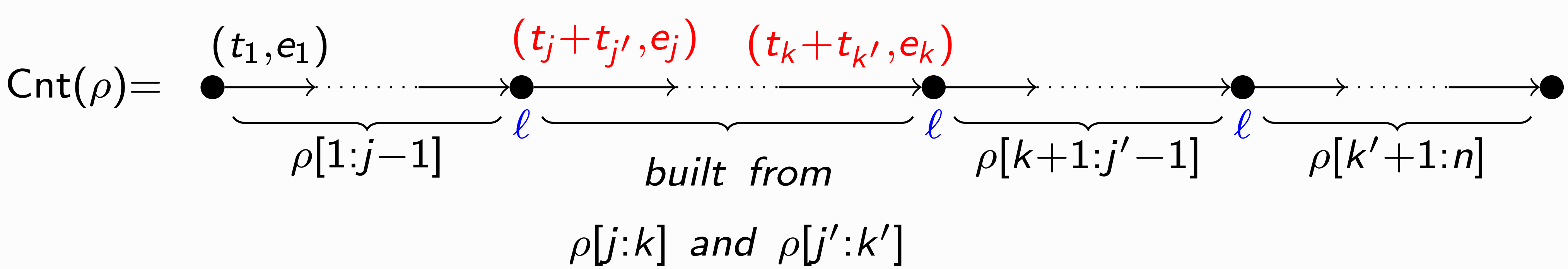
Bounding between two equalities

if big enough: cycles !

Key idea : The contraction operation

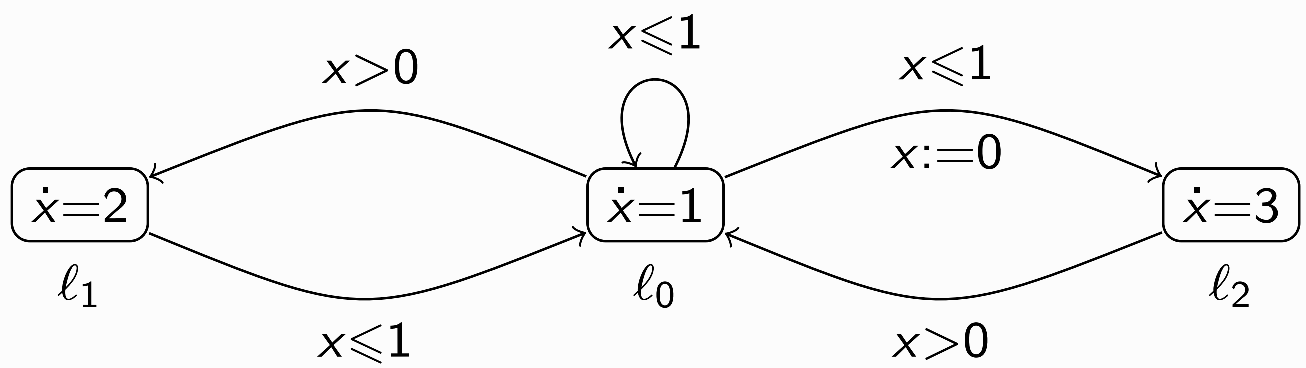


with $\rho[j:k]=\rho[j':k']$.



Bounding between two equalities

The contraction operation - A concrete example



$$\rho \;=\; (\ell_0,0) \xrightarrow{.1,e_{00}} (\ell_0,.1) \xrightarrow{.3,e_{01}} (\ell_1,.4) \xrightarrow{.1,e_{10}} (\ell_0,.6) \xrightarrow{.2,e_{00}} (\ell_0,.8) \xrightarrow{.1,e_{01}} (\ell_1,.9)$$

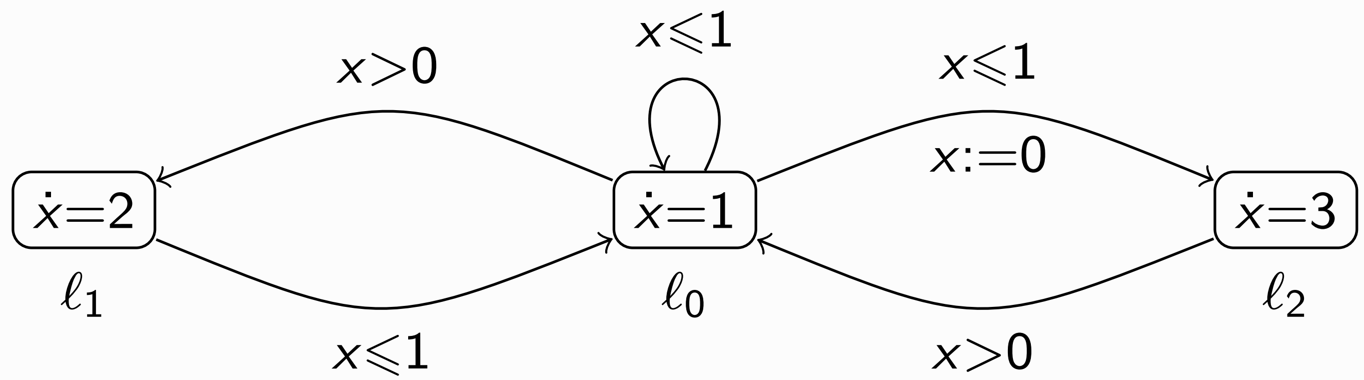
$$\mathsf{Cnt}(\rho) \;=\; (\ell_0,0) \xrightarrow{.1+.2,e_{00}} (\ell_0,.3) \xrightarrow{.3,e_{02}} (\ell_1,.6) \xrightarrow{.1,e_{20}} (\ell_0,.8) \xrightarrow{.1,e_{01}} (\ell_1,.9)$$

Advantages

- The new execution is shorter (in term of transitions).
- The value of the variables are preserved.

Bounding between two equalities

The contraction operation - Problem I



x:=0

$$\rho = (\ell_0,0) \xrightarrow{.1,e_{00}} (\ell_0,.1) \xrightarrow{.3,\mathbf{e_{02}}} (\ell_2,0) \xrightarrow{.1,e_{20}} (\ell_0,.3) \xrightarrow{.2,e_{00}} (\ell_0,.5) \xrightarrow{.4,e_{01}} (\ell_1,\mathbf{.9})$$

$$\mathsf{Cnt}(\rho) = (\ell_0,0) \xrightarrow{.1+.2,e_{00}} (\ell_0,.3) \xrightarrow{.3,e_{02}} (\ell_2,0) \xrightarrow{.1,e_{20}} (\ell_0,.3) \xrightarrow{.4,e_{01}} (\ell_1,\mathbf{.7})$$

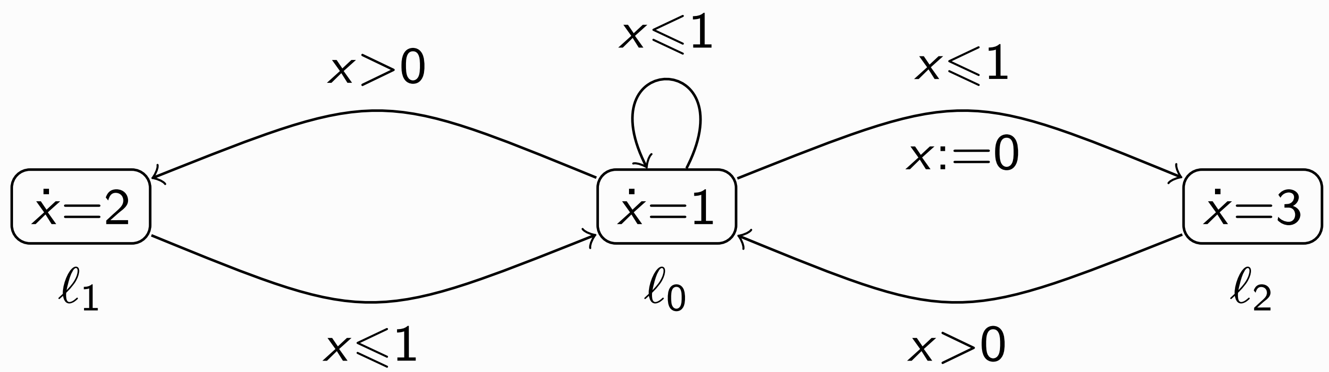
The value of the variables are not necessarily preserved...

Solution

Do not contract transitions occurring before and after the **last reset.**

Bounding between two equalities

The contraction operation - Problem II



x:=0

$$\rho = (\ell_0,0) \xrightarrow{.7,e_{00}} (\ell_0,.1) \xrightarrow{.3,e_{01}} (\ell_1,.4) \xrightarrow{.1,e_{10}} (\ell_0,.6) \xrightarrow{.2,e_{00}} (\ell_0,.8) \xrightarrow{.1,e_{01}} (\ell_1,.9)$$

$$\mathsf{Cnt}(\rho) = (\ell_0,0) \xrightarrow{.7+.2,e_{00}} (\ell_0,.9) \xrightarrow{.3,} (\ell_1,\mathbf{1.2})$$

Cnt (ρ) is not necessarily a proper execution...

Solution

- Do not contract transitions occurring before and after the **first reset**.
- Ensure that the time spent along an execution is **short enough**.

Bounding between two equalities

Building a bounded witness

Ultimate Goal

Given ρ an execution of $\mathcal{H}$ reaching ℓ_1 from (ℓ_0, x_0) within T time units.
We want to build ρ' such that :

- an execution of $\mathcal{H}$ reaching ℓ_1 from (ℓ_0, x_0) within T time units,
- the number of transitions of ρ' is bounded by a constant depending only of $\mathcal{H}$ and T.

- **Step 1 : Time-slicing**
  We can slice ρ is pieces whose duration is at most $\frac{1}{R_{max}}$.

  At most $R_{max} \cdot T$ pieces.

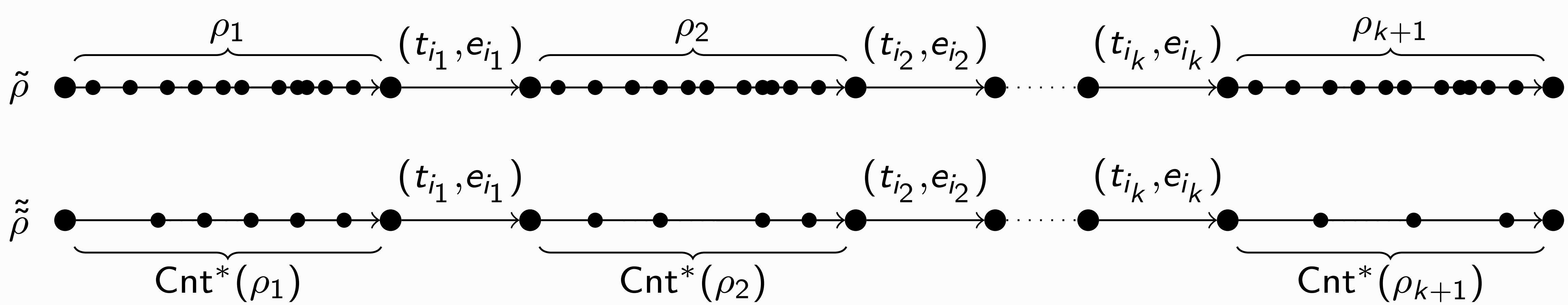
- **Step 2 : First and Last reset-slicing**
  We can slice ρ according to the first an last resets of each clock.

  At most $3 \cdot |X|$ pieces.

Bounding between two equalities

Building a bounded witness (continued)

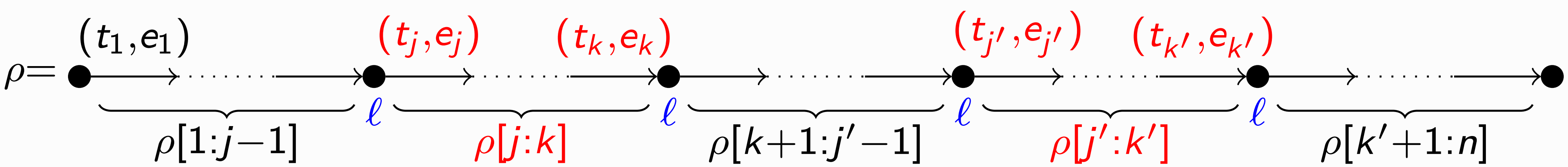
- **Step 3 : Application of the contraction :**



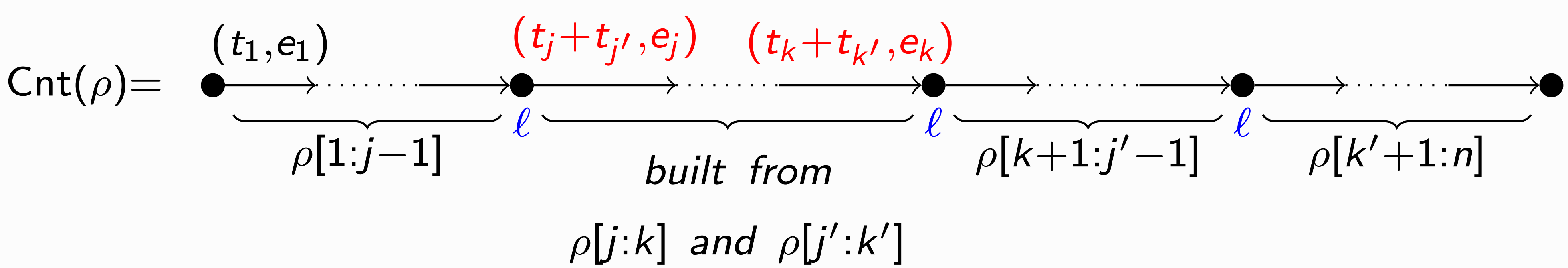
- $\tilde{\tilde{\rho}}$ is a proper execution of $\mathcal{H}$.
- The variables have the same value at the end of $\tilde{\rho}$ and $\tilde{\tilde{\rho}}$.
- The number of transitions in $\tilde{\tilde{\rho}}$ is bounded by a constant depending only of $\mathcal{H}$.

Bounding between two equalities

The contraction operation



with $\rho[j:k] = \rho[j':k']$.



$$|\mathsf{Cnt}^*(\rho)| \leq |\mathrm{Loc}| \cdot (2^{(|\mathrm{Edges}|+1)} + 1),$$

where $\mathsf{Cnt}^*(\rho)$ is the fixed point obtained by iterating $\mathsf{Cnt}(\cdot)$ to ρ.

Decision procedure for TBR

Theorem

A **Goal** location is reachable in RHA⊕ H within **T** time units

iff

it is reachable by a run ρ of **size bounded by K**(H,**T**)$\in\mathbb{N}$.

Corollary

Time bounded reachability can be reduced to the **satisfiability** of a **formula in the first order theory of the reals** encoding the existence of runs of length at most **K**(H,**T**) that reaches Goal.

Decision procedure for TBR

Theorem

A **Goal** location is reachable in RHA⊕ H within **T** time units

iff

it is reachable by a run ρ of **size bounded by K**(H,**T**)$\in\mathbb{N}$.

Corollary

Time bounded reachability can be re[redacted]
formula in the first or[redacted]
existence of [redacted]al.

Not based on a finite similarity quotient
(as it is usually the case)

Beyond RHA⊕

- Negative rates lead to **undecidability**
- Diagonal constraints lead to **undecidability**

Decidability frontier

	Reach	Time-bounded Reach
Timed automata	☝	☝
Initialized RHA	☝	☝
RHA⊕	☟ (Stopwatch)	☝
RHA	☟	☟ (neg. rates or diag.)
LHA	☟	☟
Affine HA	☟	☟
O-Minimal HA	☝	?

Conclusion

- Reachability analysis of hybrid automata have proven **useful** (embedded systems-protocols-biological systems-etc.)
- **PhaVer** and **HyTech** implements symbolic semi-algorithm for LHA-RHA
- PhaVer implements rectangular approximations of affine HA

  Details: Laurent Doyen, Tom Henzinger, Jean-François Raskin. **Automatic Rectangular Refinement of Affine Hybrid Systems**. In FORMATS'05, Lecture Notes in Computer Science 3829, pp. 144--161, Springer-Verlag, 2005.
- **Time-bounded** reachability is **decidable** for RHA⊕ (⊇stopwatch HA)

  Details: Thomas Brihaye, Gilles Geeraerts, Laurent Doyen, Joel Ouaknine, Jean-François Raskin and James Worrell. **On reachability for Hybrid Automata over Bounded Time**. In ICALP'11, LNCS 6756, Springer, pp. 416-427, 2011.