Complexity Analysis of the Backward Coverability Algorithm for VASS

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► RP '11, Genova ◄

Safety checking for systems with infinitely many states

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Low supply of decidable cases, e.g. coverability checking for VASS

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Coverability checking

Vector Addition Systems with States

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Coverability checking for VASS is useful for verification:

- Multithreaded Software Libraries (Ball, Chaki, Rajamani in TACAS '01)
- Asynchronous programs

(Majumdar, Jhala, Sen, Viswanathan in POPL '{07,09}, CAV '06, TCS '09)

- Parameterized Concurrent Programs

(Kroening, Kaiser, Wahl in CAV '10)

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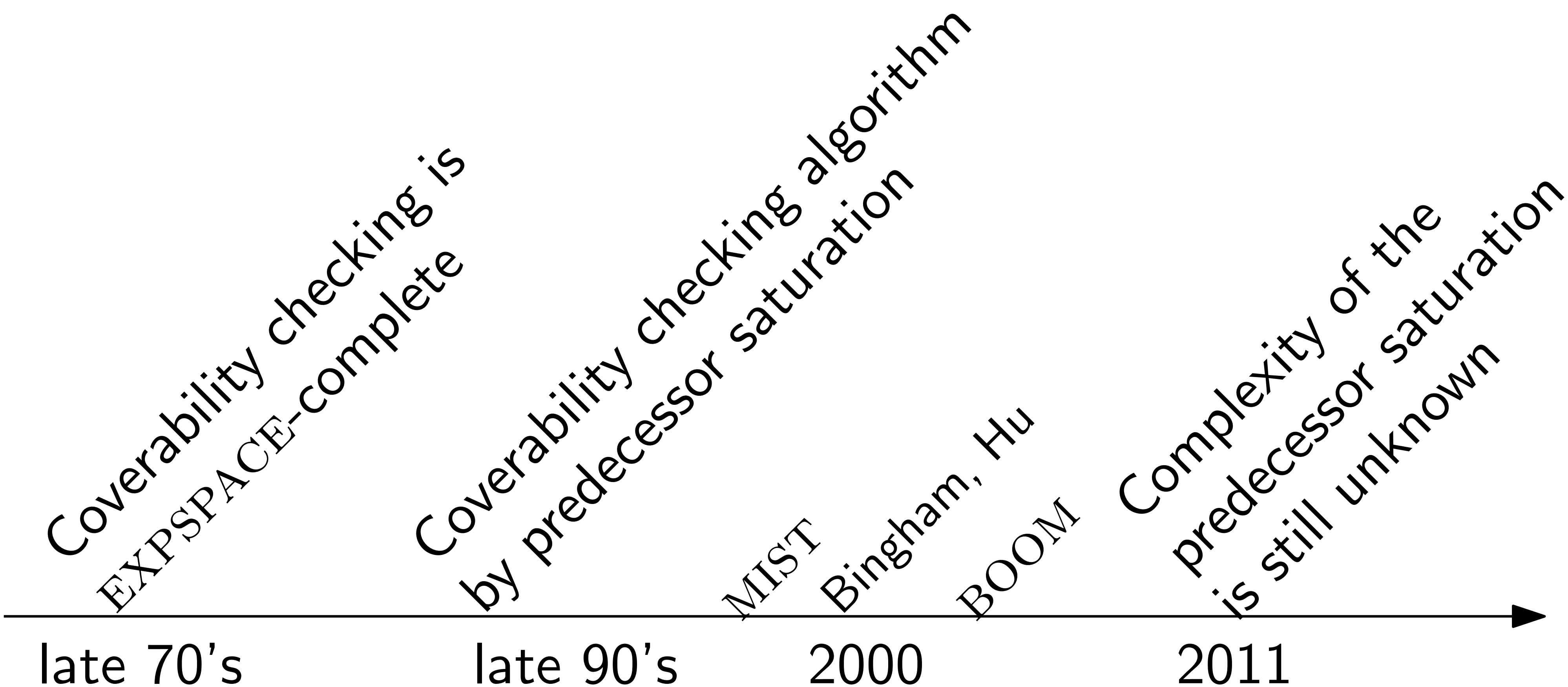
VASS coverability checker have been implemented:
- Delzanno, Raskin, Van Begin, G (MIST, 2000–2007)
- Bingham, Hu (2005)
- Kaiser, Kroening, Wahl (BOOM, 2009–present)

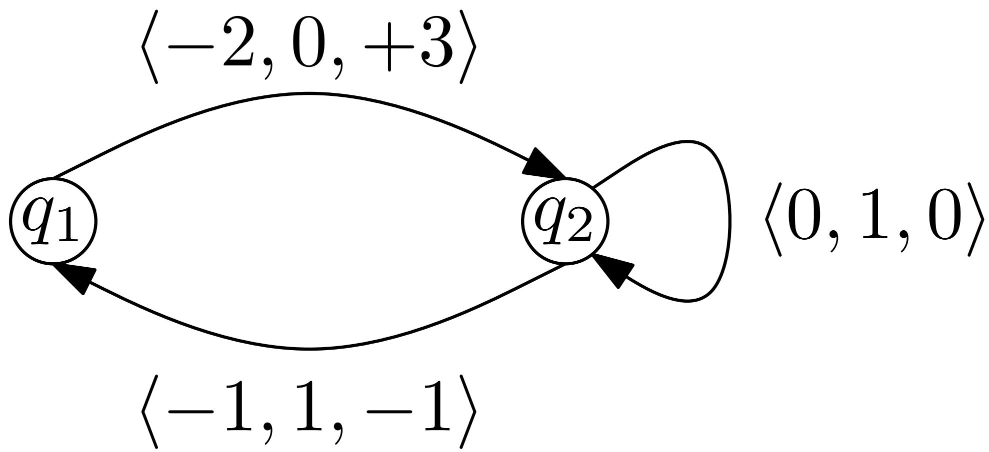
Complexity of problem vs solution



VASS

Example of a three dimensional VASS, or 3-VASS for short:

Finite state automaton over finite alphabet taken in $\mathbb{Z}^3$



VASS

A d-VASS consists of a pair (Q, Δ)

- Q are the control states
- $\Delta \subseteq Q \times \mathbb{Z}^d \times Q$ is the finite set of transitions

Semantics as an infinite transition system $(Q \times \mathbb{N}^d, \rightarrow)$

control state of the fsa $\in Q$

$$\langle q, v_1, \ldots, v_d \rangle$$

counter valuation $\in \mathbb{N}$

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Semantics as an infinite transition system $(Q \times \mathbb{N}^d, \rightarrow)$

Let $q \xrightarrow{\vec{u}} q'$ be a transition where $\vec{u} = \langle u_1, \ldots, u_d \rangle$ then

$$\langle q, v_1, \ldots, v_d \rangle \rightarrow \langle q', v_1 + u_1, \ldots, v_d + u_d \rangle$$

- $\rightarrow^*$ denotes the reachability relation

VASS coverability

Define the ordering $\trianglelefteq$ on VASS states as follows:

$$\langle q, v_1, \ldots, v_d \rangle \trianglelefteq \langle q', v_1', \ldots, v_d' \rangle$$

iff

$$q = q' \text{ and } v_i \leq v_i' \text{ for every } i \in \{1, \ldots, d\}$$

$\boldsymbol{s}_\imath \circ$ $\circ\ \boldsymbol{s}_f$

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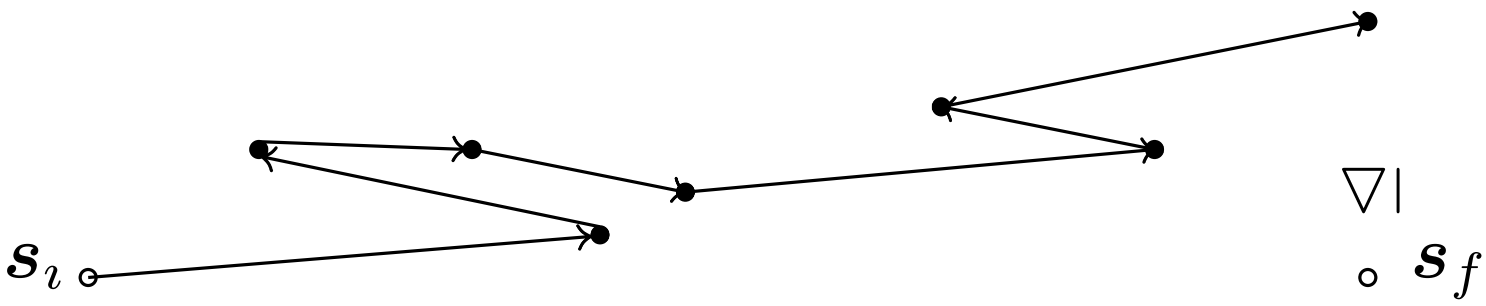

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Given a VASS and two VASS-states: $\boldsymbol{s}_\imath$ and $\boldsymbol{s}_f$
Checking that $\boldsymbol{s}_f$ is coverable from $\boldsymbol{s}_\imath$ asks
if there exists a VASS-state $\boldsymbol{s}$ such that:

$$\boldsymbol{s}_\imath \rightarrow^* \boldsymbol{s} \text{ and } \boldsymbol{s}_f \trianglelefteq \boldsymbol{s}$$

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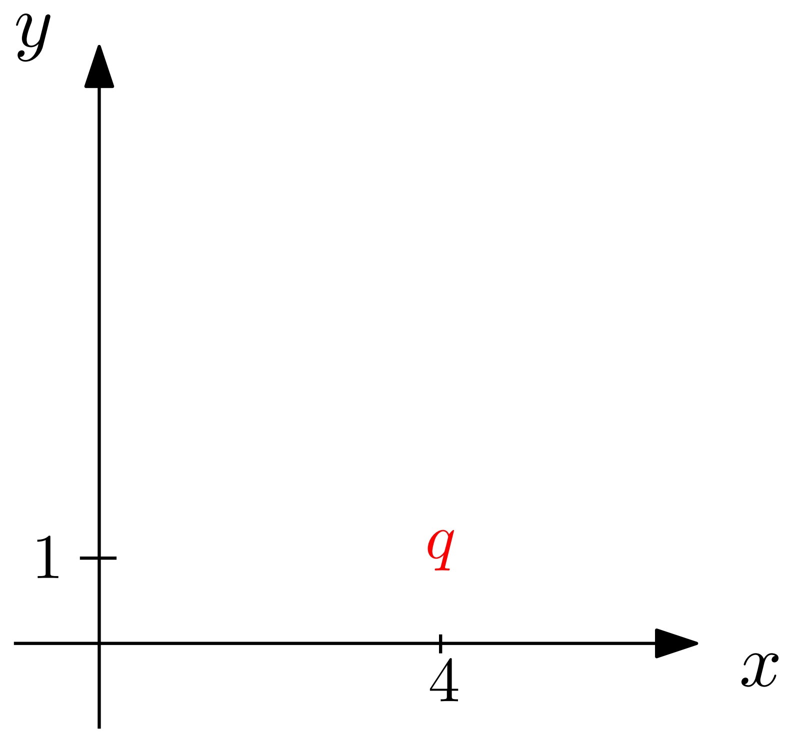
Seminal papers about the coverability problem:

- Lower bound on the coverability problem, Lipton 1976, cited 219 times
- Upper bound on the coverability problem, Rackoff 1978, cited 143 times

The coverability problem is EXPSPACE-complete

Solving VASS coverability: predecessors computation

Let a 2-VASS and call the 2 counters $\{x, y\}$ and $\boldsymbol{s}_f = \langle q, 4, 1 \rangle$



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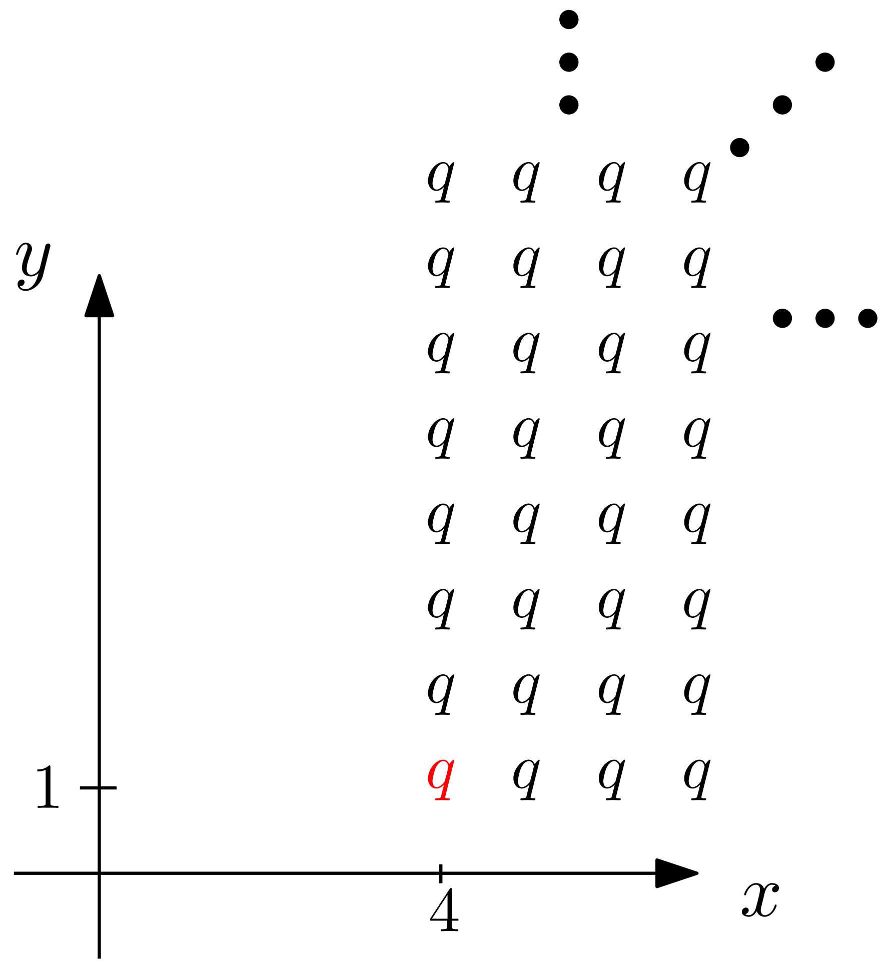

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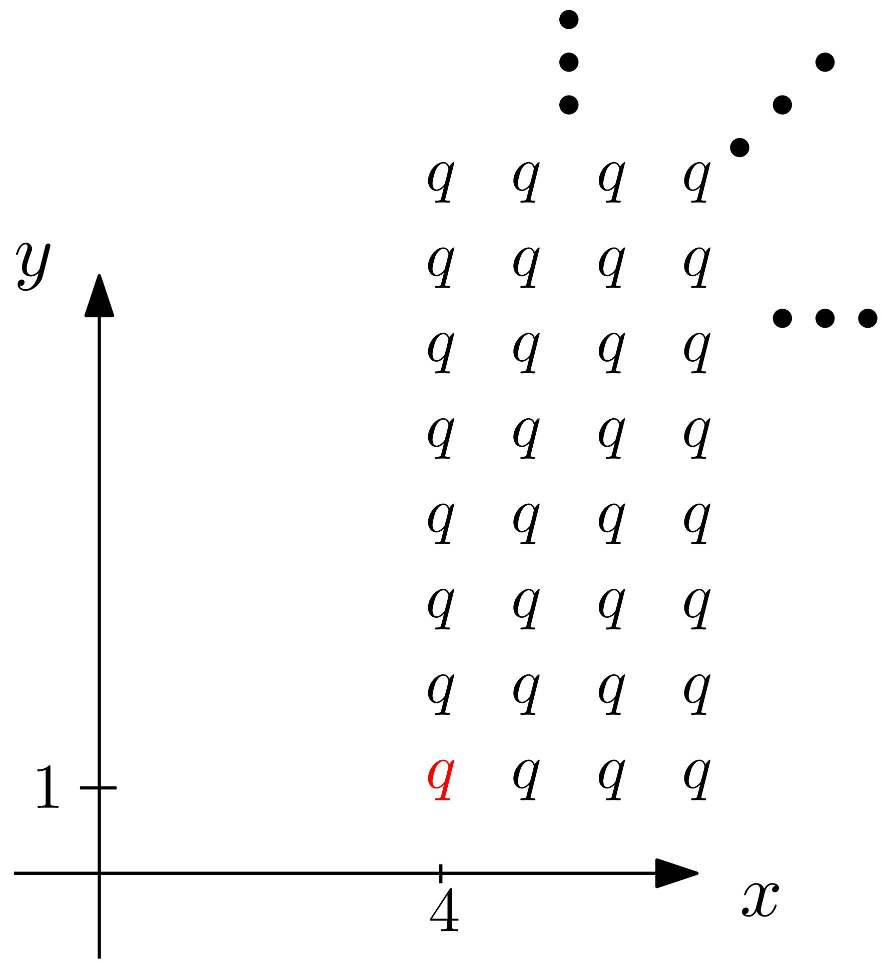

Let $t\colon q_1 \xrightarrow{\langle +1,0 \rangle} q$

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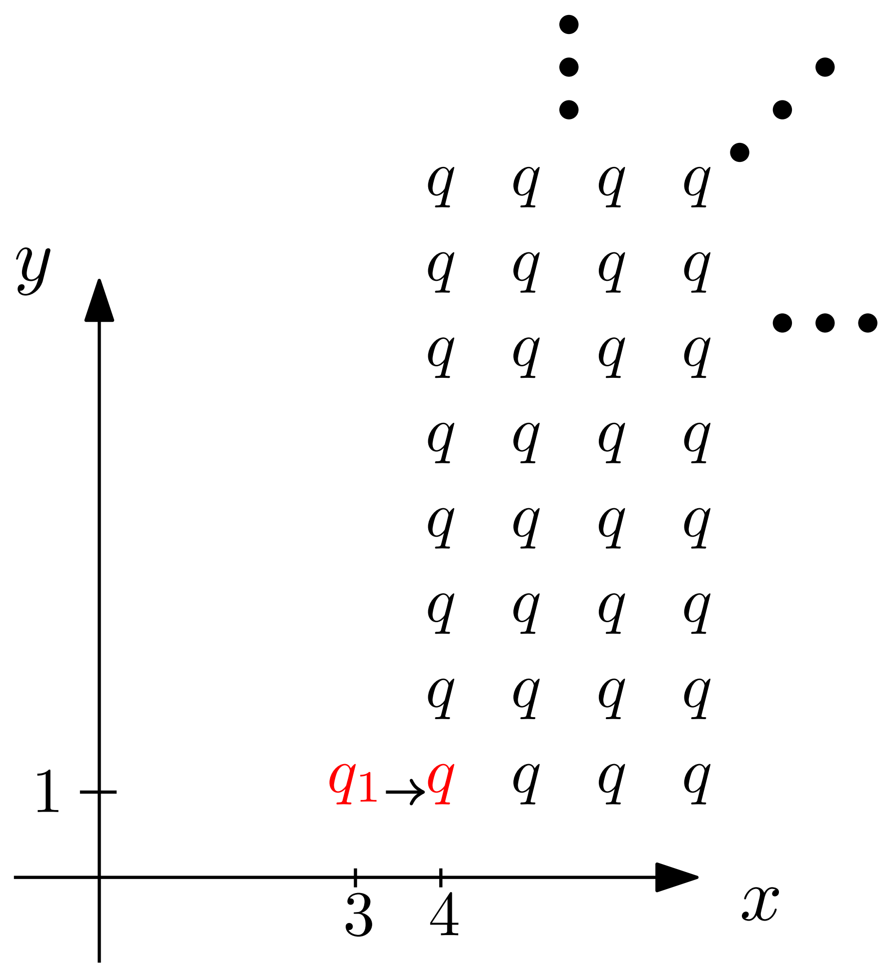

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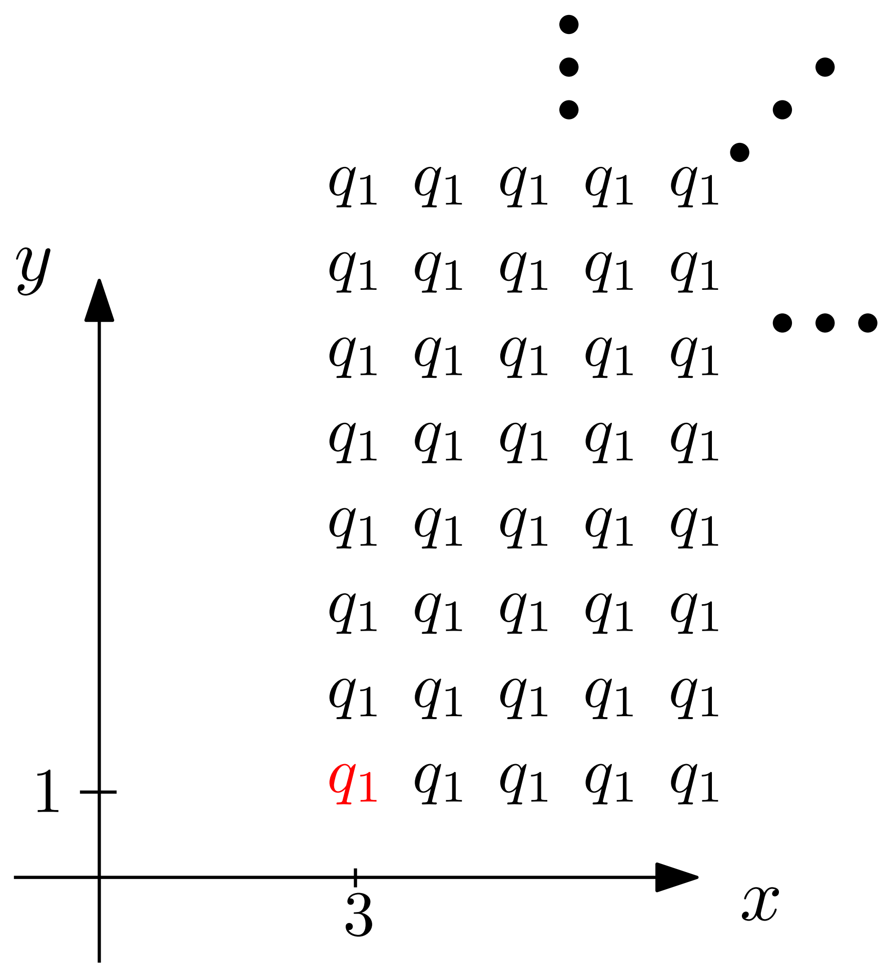

Let $t\colon q_1 \xrightarrow{\langle +1,0\rangle} q$

$pre[t](\{\langle q, i, j\rangle \mid i \geq 4, j \geq 1\}) = \{\langle q_1, k, \ell\rangle \mid k \geq 3, \ell \geq 1\}$

Solving VASS coverability

Saturating the predecessor computation

$pre[t](X)$ predecessor of X in 1 step using t

$pre(X) = \bigcup_t pre[t](X)$ predecessor in 1 step

$pre^*(X)$ predecessors in 0 or more steps

$$pre^*(U) = \lim_{n\to\infty} X_n \quad \text{where} \begin{cases} X_1 = U \\ X_{i+1} = U \cup pre(X_i) \end{cases}$$

$$X_1 = U$$

$$X_2 = U \cup pre(U)$$

$$X_3 = U \cup pre(U) \cup pre(pre(U))$$

$$\vdots$$

$\boldsymbol{s}_f$ is coverable from $\boldsymbol{s}_\imath$ iff $\boldsymbol{s}_\imath \in pre^*(\{\boldsymbol{s} \mid \boldsymbol{s}_f \trianglelefteq \boldsymbol{s}\})$

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$$\begin{array}{l} X_1 = U \\ \rotatebox{90}{$\subseteq$} \\ X_2 = U \cup pre(U) \\ \rotatebox{90}{$\subseteq$} \\ X_3 = U \cup pre(U) \cup pre(pre(U)) \\ \vdots \end{array}$$

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$pre^*(U)$ convergence and effectiveness

Convergence: If U is $\trianglelefteq$-closed then $X_1, X_2, \ldots$ is such that

- X_i is $\trianglelefteq$-closed for every i
- stabilizes after finitely many steps: $X_\dagger = X_{\dagger+1}$ for some $\dagger$

Effectiveness: Let U $\trianglelefteq$-closed, $\min(U)$ finitely represents U.

- Predicates $U_1 \subseteq U_2$ and $\boldsymbol{s}_\imath \in U$ for U, U_1, U_2 $\trianglelefteq$-closed are decidable given $\min(U), \min(U_1), \min(U_2)$.
- $\min(pre(U))$ is finite and computable given $\min(U)$

Let

$$Z_1 = \min(U)$$

$$Z_{i+1} = \min\big(\min(U) \cup minpre(Z_i)\big)$$

then $Z_i = \min(X_i)$ for every i

pr ctiveness

$\boldsymbol{s} \in U \wedge \boldsymbol{s} \trianglelefteq \boldsymbol{s'}$ implies $\boldsymbol{s'} \in U$

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$$\boldsymbol{s}_f \text{ is coverable from } \boldsymbol{s}_\imath \text{ iff } \begin{cases} \boldsymbol{s}_\imath \in pre^*(\{s_f \mid \boldsymbol{s}_f \trianglelefteq s\}) \\ \boldsymbol{s} \trianglelefteq \boldsymbol{s}_\imath \text{ for some } \boldsymbol{s} \in Z_\dagger \end{cases}$$

Upper Bound

Complexity of the predecessor algorithm: upper bounds

Given d-VASS $G = (Q, \Delta)$ and G-state $\boldsymbol{s}_f$

Let

$$Z_1 = \{\boldsymbol{s}_f\}$$
$$Z_{i+1} = \min\big(\{\boldsymbol{s}_f\} \cup \mathit{minpre}(Z_i)\big)$$

From the EXPSPACE membership proof we derive:

- upper bound on $\dagger$
- upper bound on $|Z_i|$ for every i
- upper bound b such that $Z_i \subseteq Q \times [0, b]^d$ for every i
- hence, upper bound on the execution time

Define $b_c \in \mathbb{N}_0$ be the least value such that
$$\Delta \subseteq Q \times [-b_c, +b_c]^d \times Q \text{ and } \boldsymbol{s}_f \in Q \times [0, b_c]^d$$

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- upper bound on $|Z_i|$ for every i $\boxed{O(|Q| \cdot ((i+1) \cdot b_c)^d)}$
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Rackoff's proof

A d-VASS $G = \langle Q, \Delta \rangle$ and a G-state $\boldsymbol{s}_f$

Define $\pi_{\boldsymbol{s}}$ as a shortest sequence which covers $\boldsymbol{s}_f$ from $\boldsymbol{s}$

Given G and $\boldsymbol{s}_f$, Rackoff bounds $\max_{\boldsymbol{s}} |\pi_{\boldsymbol{s}}|$

The bound is obtained by induction on the number of counters:

$$f(0) \leq f(1) \leq \cdots \leq f(d) = \max_{\boldsymbol{s}} |\pi_{\boldsymbol{s}}|$$

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$i = 0$

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Define $f(0) = \max_{\boldsymbol{s}} |\pi_{\boldsymbol{s}}|$, then $f(0)$ is bounded by $|Q|$

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$$\boldsymbol{s}_f = \left\langle q_f, v_1', \ldots, v_i', v_{i+1}', \ldots, v_d' \right\rangle$$

Let $(q \xrightarrow{\vec{u}} q') \in \Delta$ where $\vec{u} = \langle u_1, \ldots, u_i, u_{i+1}, \ldots, u_d \rangle$ then

$$\langle q, v_1, \ldots, v_i, v_{i+1}, \ldots, v_d \rangle$$

$$\rightarrow$$

$$\langle q', v_1 + u_1, \ldots, v_i + u_i, v_{i+1} + u_{i+1}, \ldots, v_d + u_d \rangle$$

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$$f(0) \leq f(1) \leq \cdots \leq f(d) = \max_{\boldsymbol{s}} |\pi_{\boldsymbol{s}}|$$

$i > 0$

$$\boldsymbol{s}_f = \langle q_f, v'_1, \ldots, v'_i, \underset{0}{\cancel{v'_{i+1}}}, \ldots, \underset{0}{\cancel{v'_d}} \rangle$$

Let $(q \xrightarrow{\vec{u}} q') \in \Delta$ where $\vec{u} = \langle u_1, \ldots, u_i, \overset{0}{\cancel{u_{i+1}}}, \ldots, \overset{0}{\cancel{u_d}} \rangle$ then

$$\langle q, v_1, \ldots, v_i, v_{i+1}, \ldots, v_d \rangle$$

$$\rightarrow$$

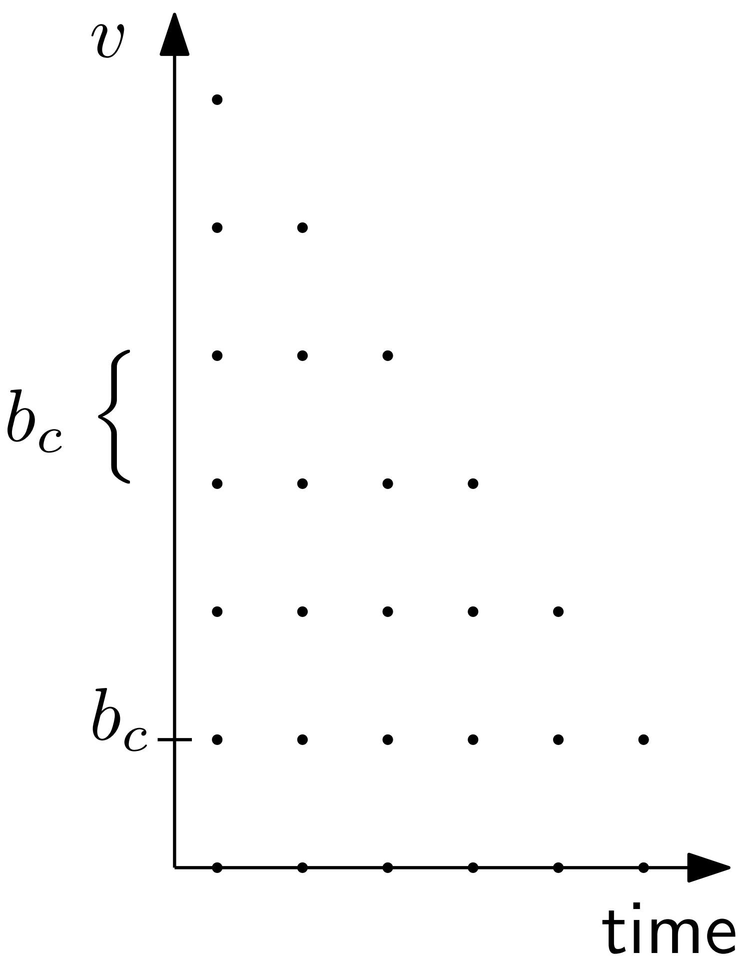
$$\langle q', v_1 + u_1, \ldots, v_i + u_i, v_{i+1} + \underset{0}{\cancel{u_{i+1}}}, \ldots, v_d + \underset{0}{\cancel{u_d}} \rangle$$

We will now characterize $f(i)$ using $f(i-1)$

Characterizing $f(i)$ using $f(i-1)$

Recall $b_c \in \mathbb{N}_0$ is the least value such that

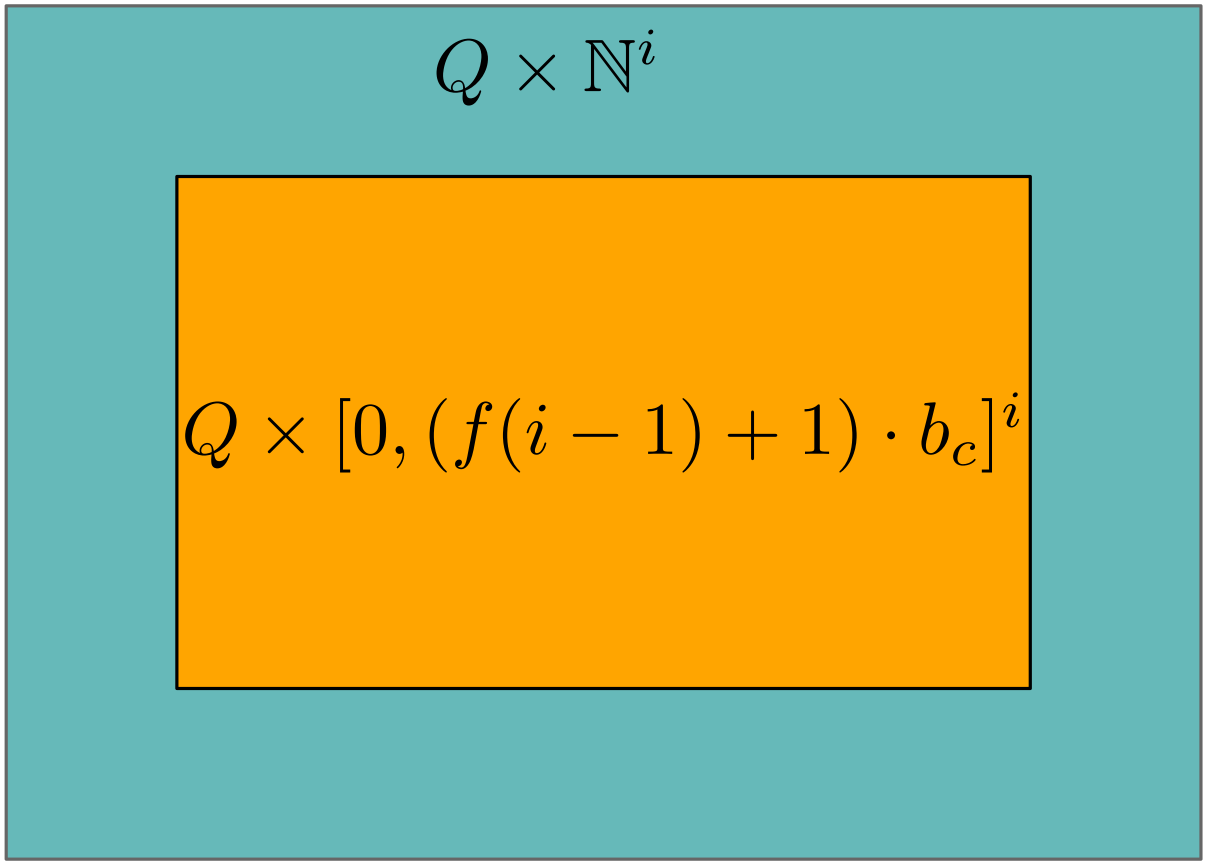
$$\Delta \subseteq Q \times [-b_c, +b_c]^d \times Q \text{ and } \boldsymbol{s}_f \in Q \times [0, b_c]^d$$



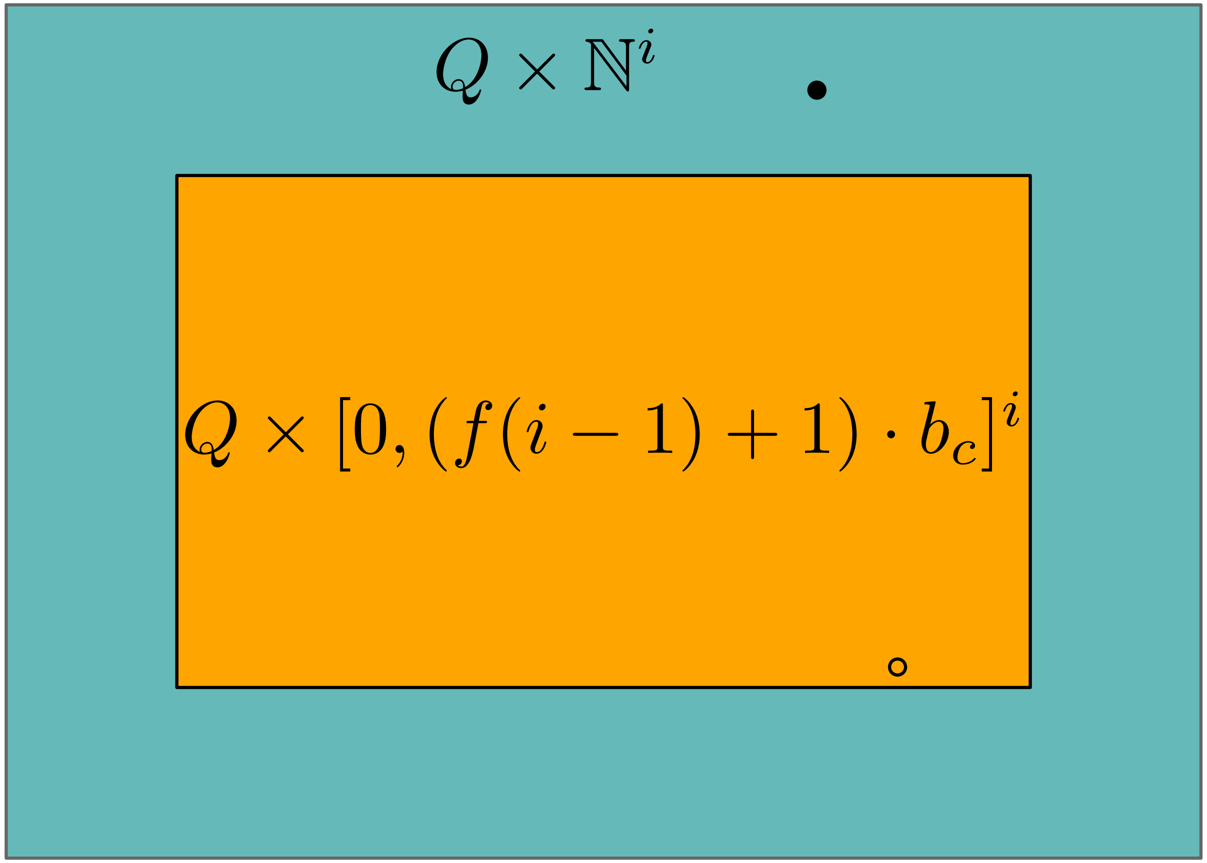
- If $v \geq 6 \cdot b_c$ then after firing any sequence π of transitions such that $|\pi| \leq 5$, then $v \geq b_c$
- If $v \geq (f(i-1)+1) \cdot b_c$ then any sequence π of transitions such that $|\pi| \leq f(i-1)$ yields $v \geq b_c$
- If some counter ever goes above $(f(i-1)+1) \cdot b_c$ then we are in the $i-1$ case

The induction hypothesis appears

Characterizing $f(i)$ using $f(i-1)$ (cont'd)



Characterizing $f(i)$ using $f(i-1)$ (cont'd)



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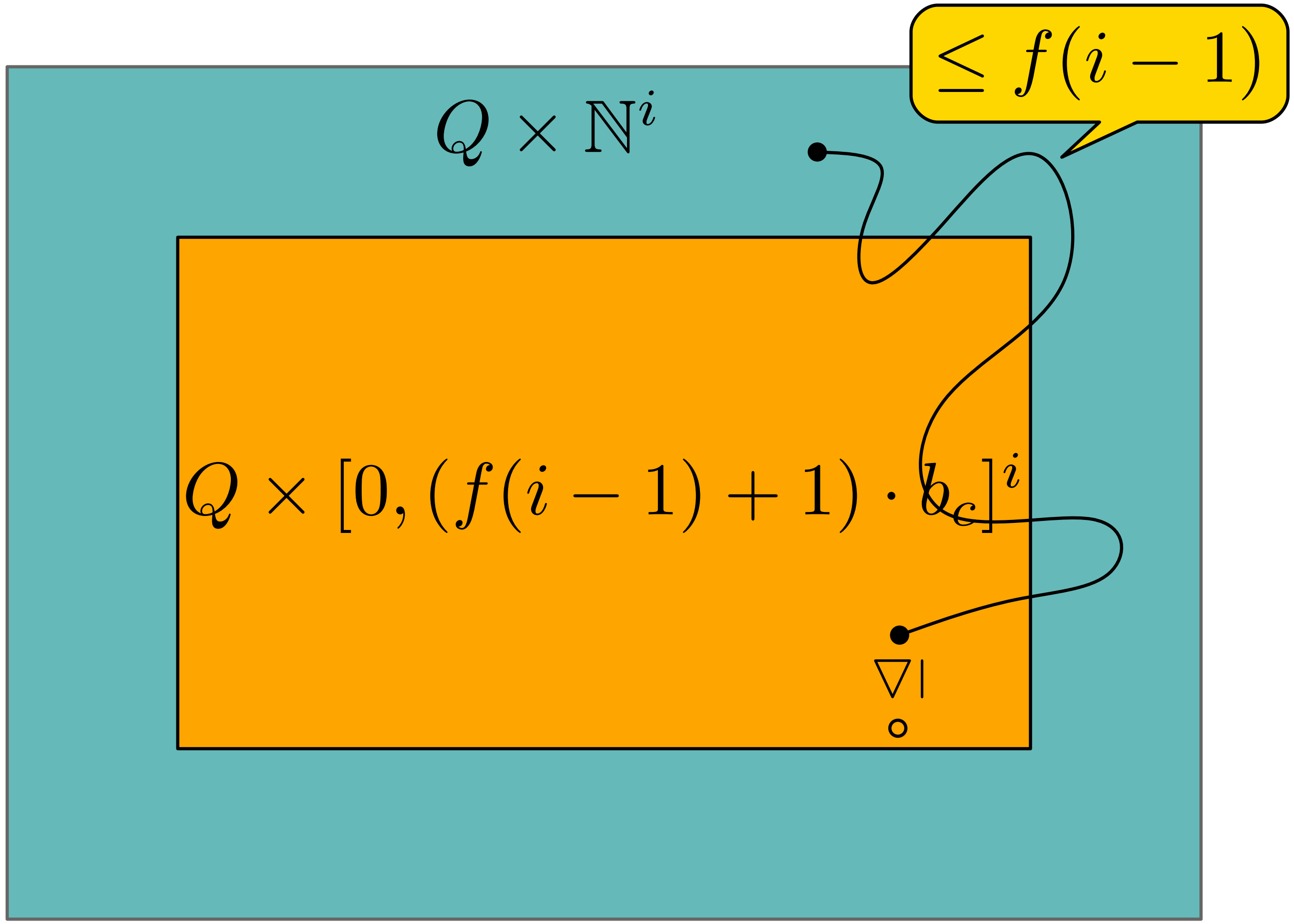
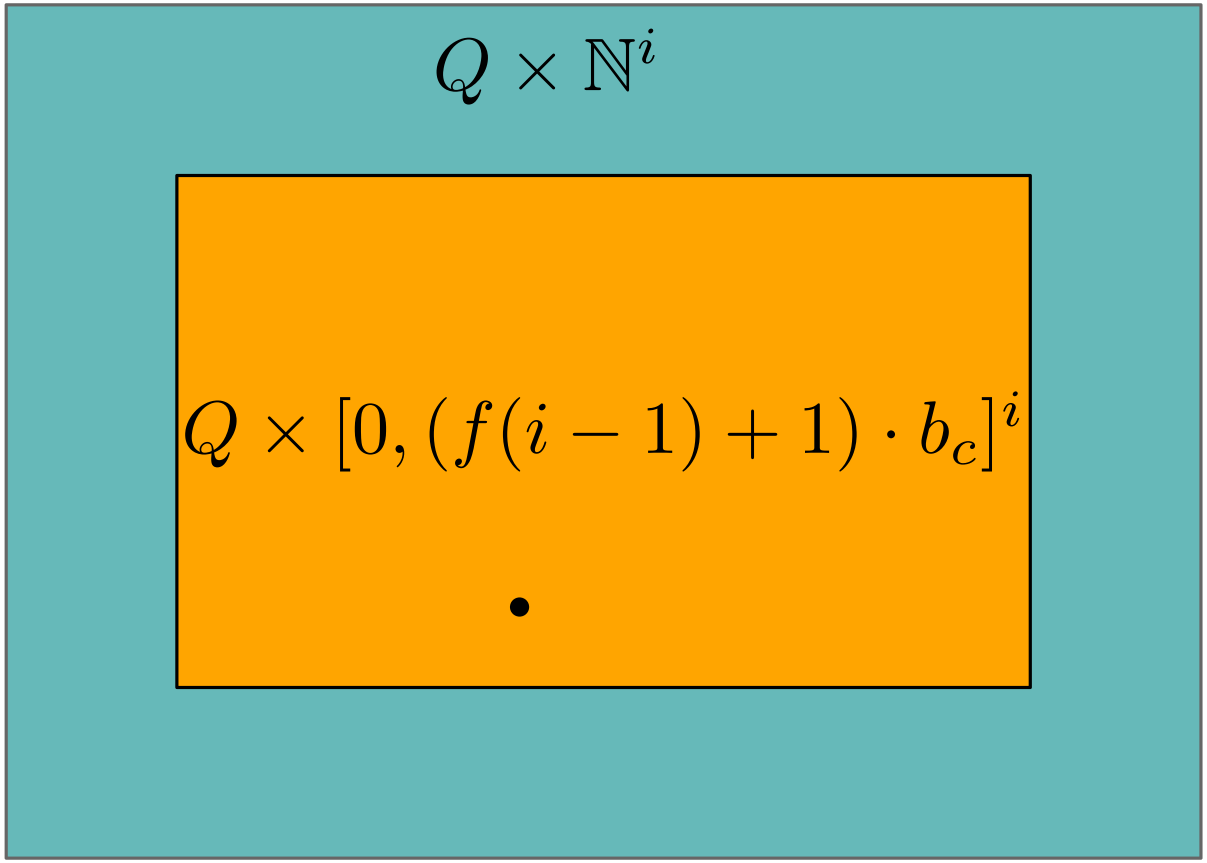


Characterizing $f(i)$ using $f(i-1)$ (cont'd)



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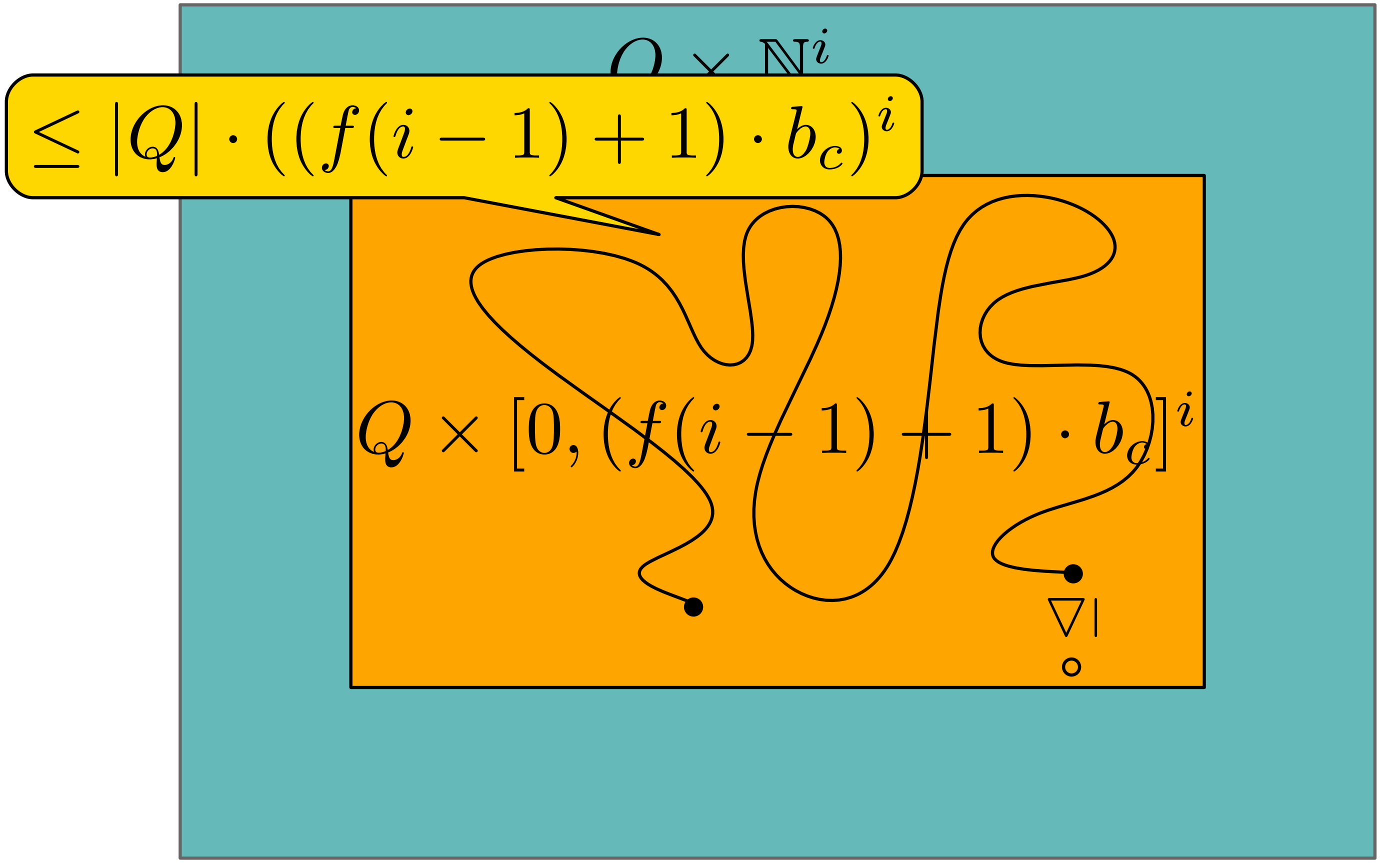
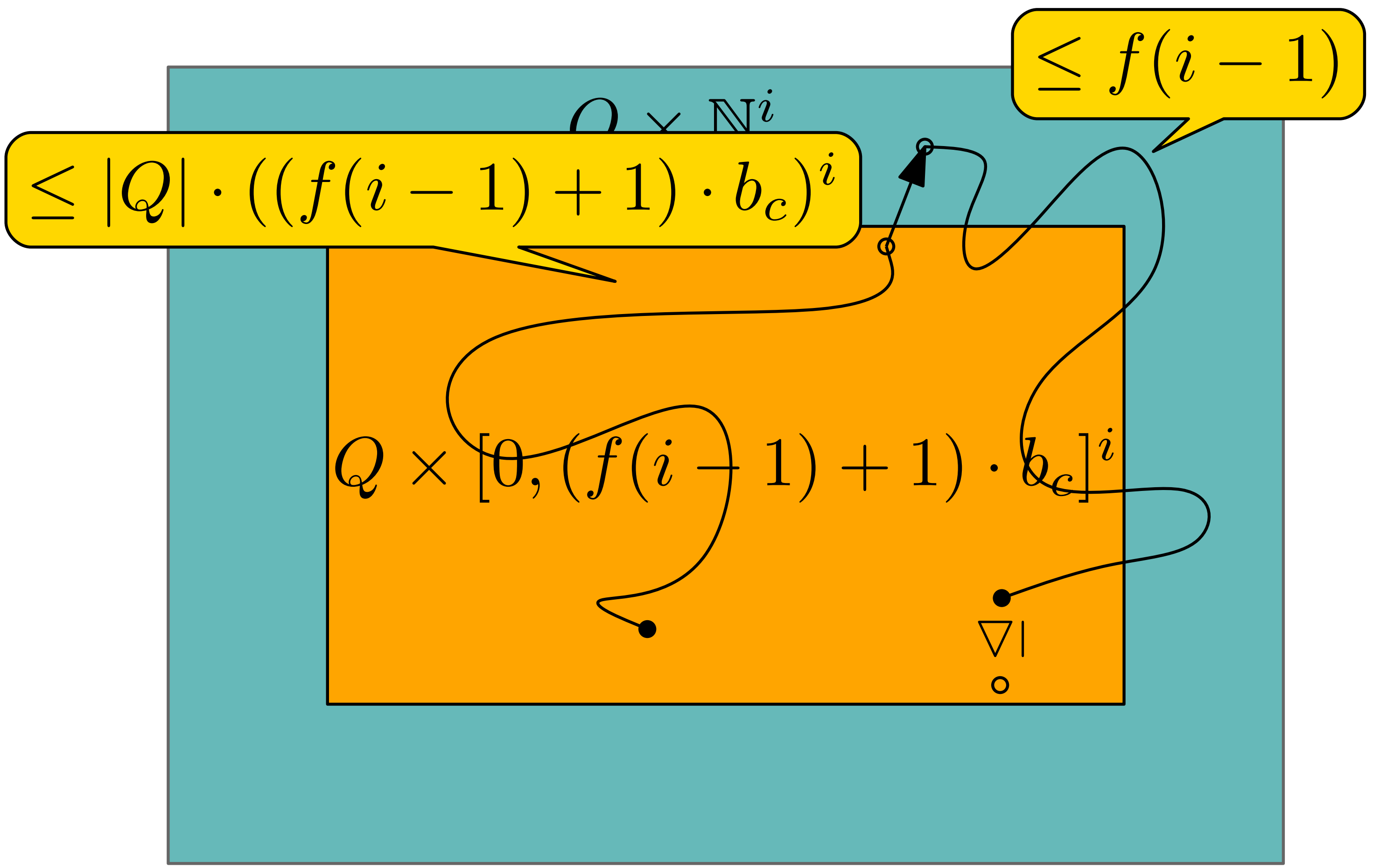


Characterizing $f(i)$ using $f(i-1)$ (cont'd)



- $f(i) \leq |Q| \cdot ((f(i-1)+1) \cdot b_c)^i + f(i-1)$
- Hence we can show that: $f(d) \leq \left(|Q| \cdot b_c\right)^{2^{O(d \cdot \log d)}}$

Back to the predecessor algorithm

Given d-VASS $G = (Q, \Delta)$ and G-state $\boldsymbol{s}_f$

Let

$$Z_1 = \{\boldsymbol{s}_f\}$$

$$Z_{i+1} = \min\bigl(\{\boldsymbol{s}_f\} \cup minpre(Z_i)\bigr)$$

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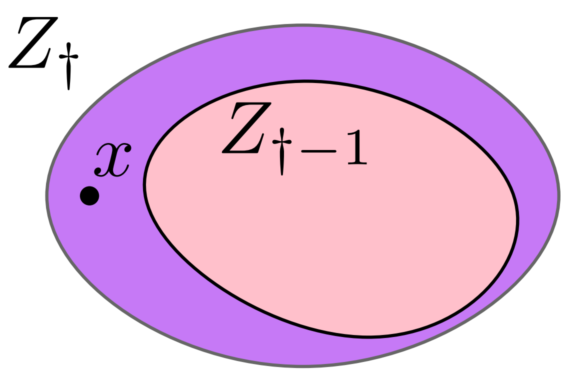
Recall:

$\pi_{\boldsymbol{s}}$ is a shortest run to cover $\boldsymbol{s}_f$ from $\boldsymbol{s}$ and $f(d) \geq \max_{\boldsymbol{s}} |\pi_{\boldsymbol{s}}|$

Suppose $\dagger > f(d)$. From there we conclude $\dagger > \max_{\boldsymbol{s}} |\pi_{\boldsymbol{s}}|$



Z_i = the states covering $\boldsymbol{s}_f$ in at most i steps

x covers $\boldsymbol{s}_f$ in no less than $\dagger$ steps, hence $|\pi_{\boldsymbol{x}}| > \max_{\boldsymbol{s}} |\pi_{\boldsymbol{s}}|$

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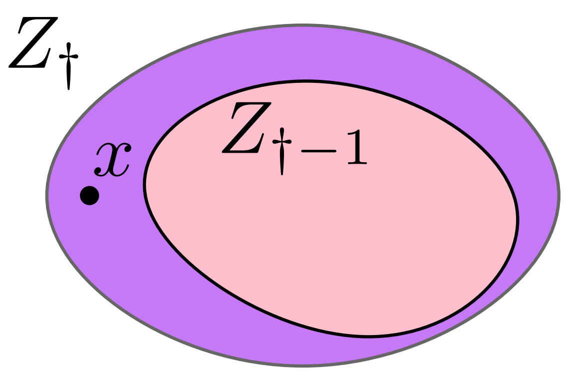

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Therefore $\dagger \leq f(d)$ and $\dagger = \big(|Q| \cdot b_c\big)^{2^{O(d \cdot \log d)}}$

Lower Bound

Complexity of the predecessor algorithm: lower bounds

Lipton's EXPSPACE-hardness result for reachability in VASS ...
defines a family $\{(G_i, \langle q_i, 0, \ldots, 0\rangle)\}_{i \in \mathbb{N}_0}$ of VASS+G_i-state for which the sequence $Z_1, Z_2, \ldots$ given by

$$Z_1 = \{\langle q_i, 0, \ldots, 0\rangle\}$$
$$Z_{j+1} = \min\big(\{\langle q_i, 0, \ldots, 0\rangle\} \cup \mathit{minpre}(Z_j)\big)$$

is such that:

- $\dagger_i \geq 2^{2^i}$
- $|Z_{\dagger_i}| \geq 2^{2^i}$
- the highest number in $Z_{\dagger_i}$ is at least $2^{2^{\Omega(i)}}$

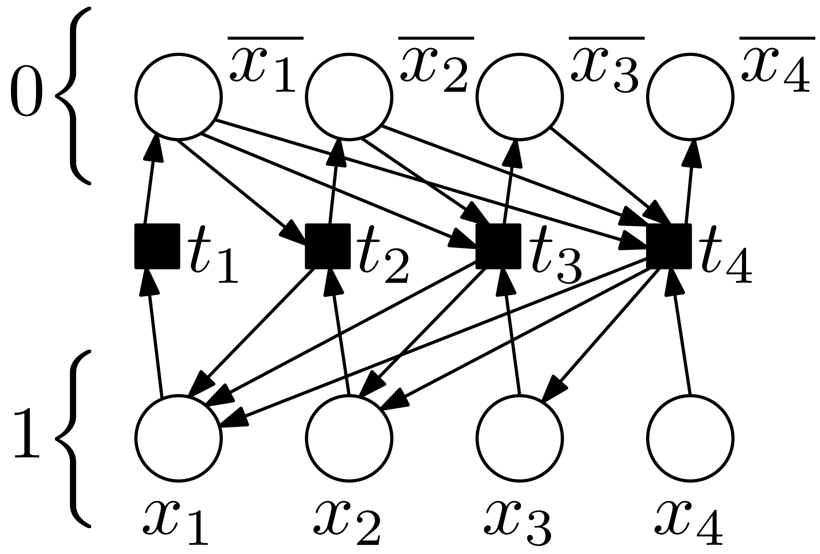
Each d-VASS $G_i = \langle Q_i, \Delta_i\rangle$ is such that:

- $d = O(i)$
- $|Q_i| = O(i)$
- $|\Delta_i| = O(i)$
- $\Delta_i \subseteq Q_i \times [-1, 1]^d \times Q_i$

Proof ideas

We want:
- a "large" value for $\dagger$
- "many" (incomparable) elements in $Z_\dagger$.

First idea, a binary counter:



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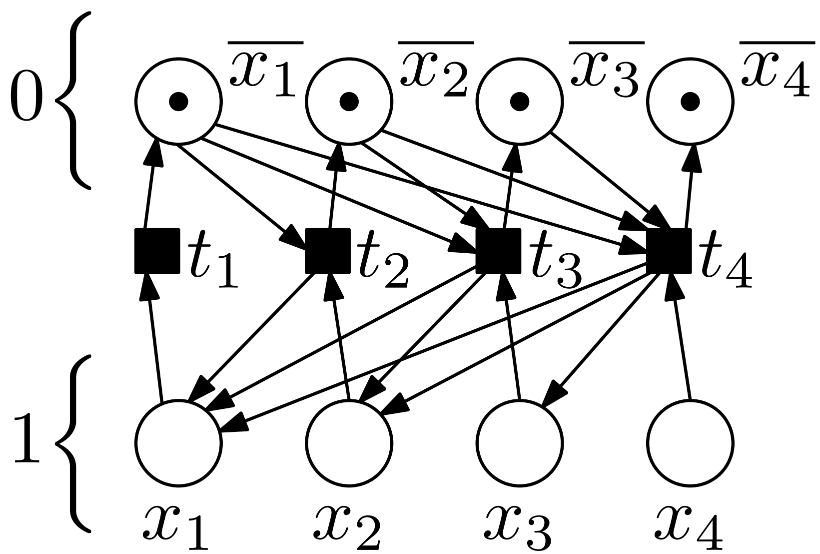

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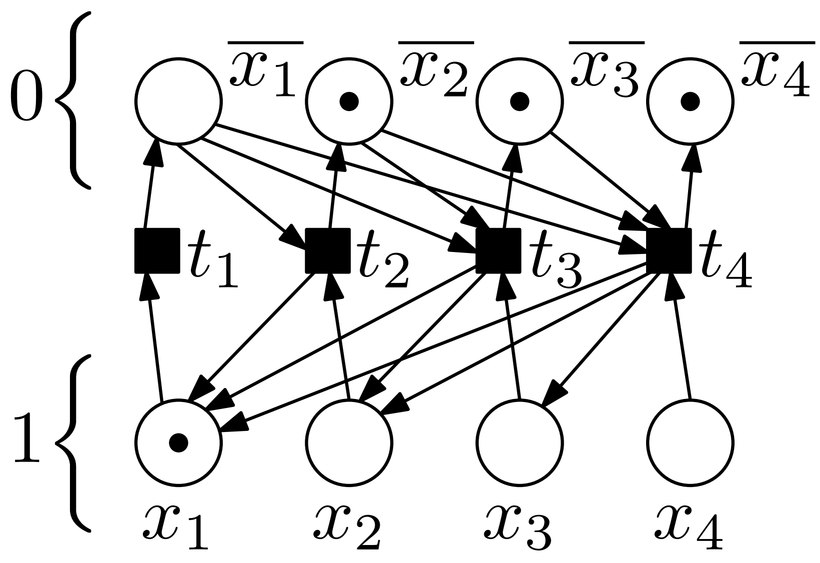

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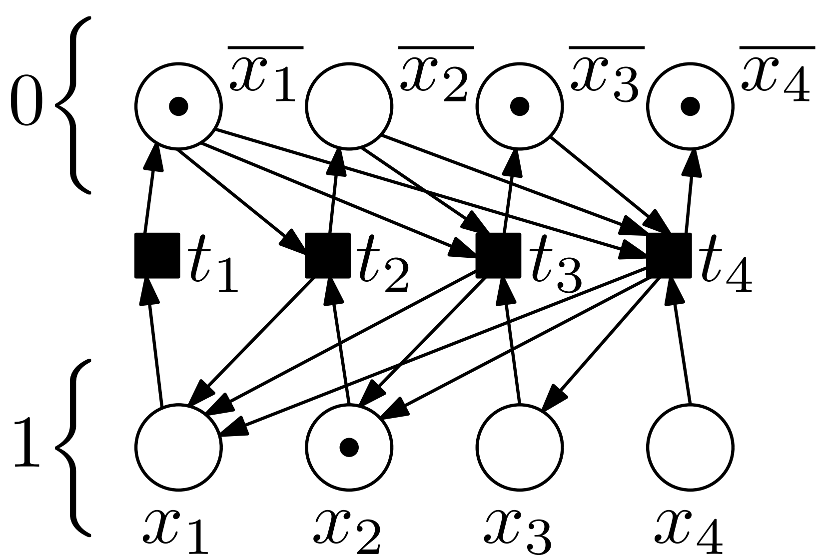

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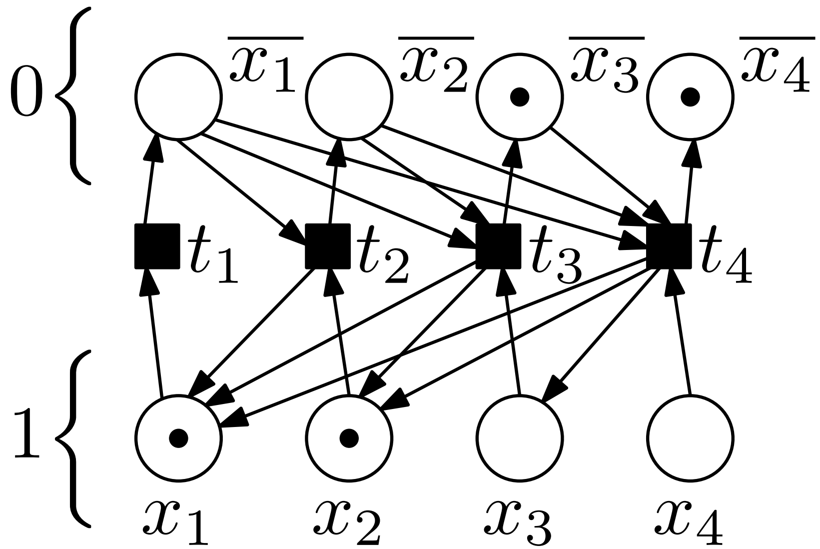

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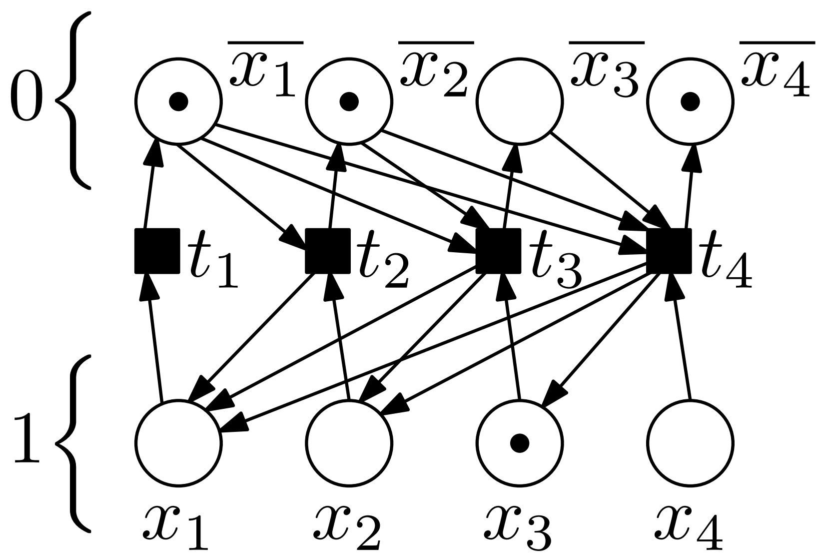

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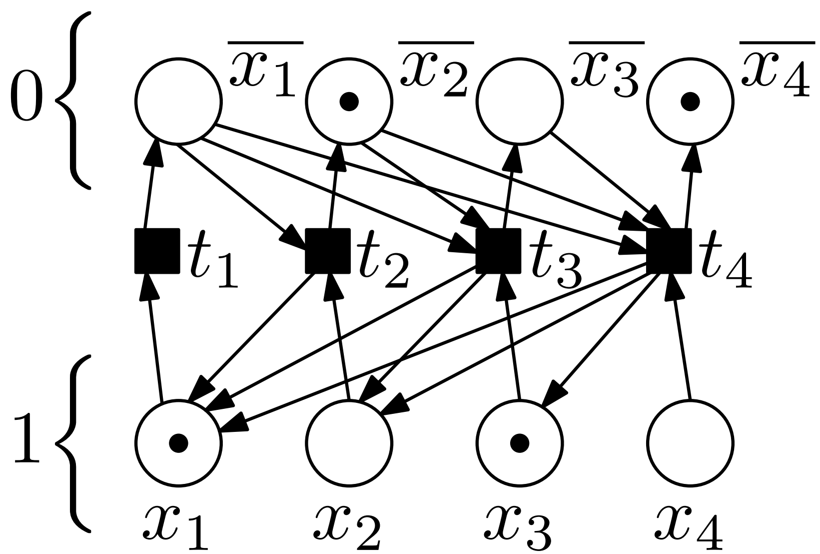

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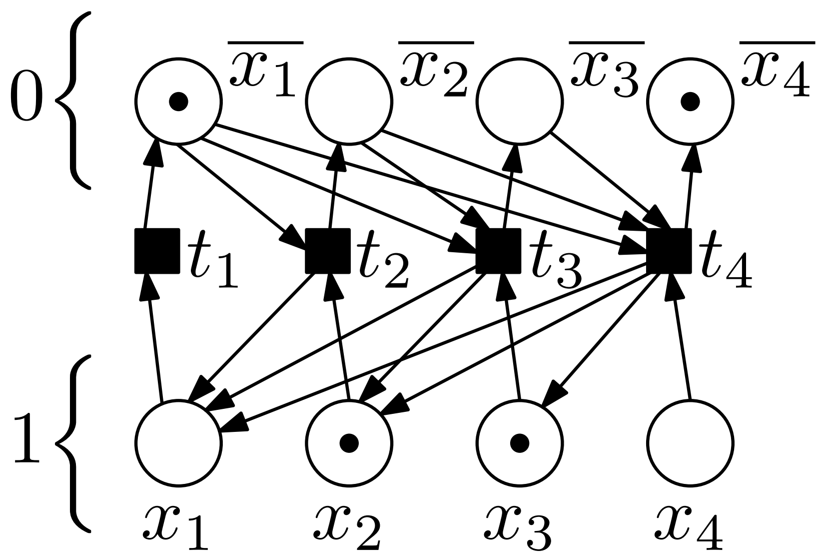

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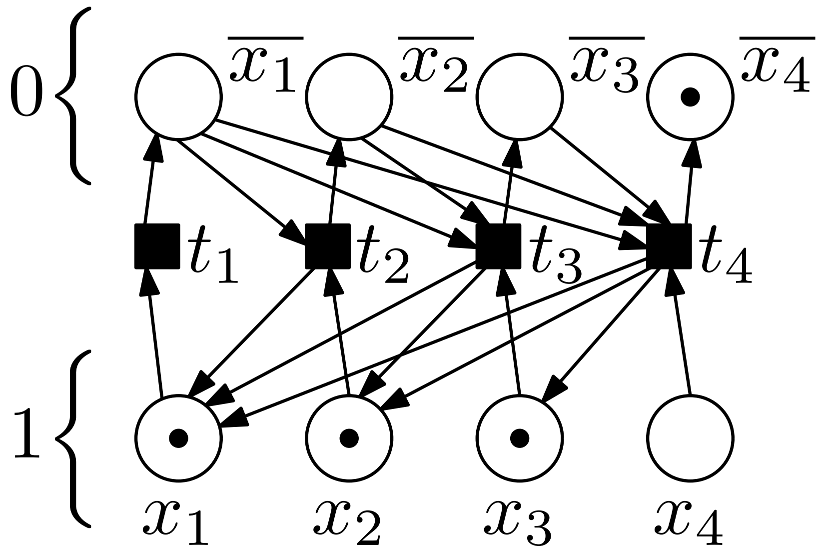

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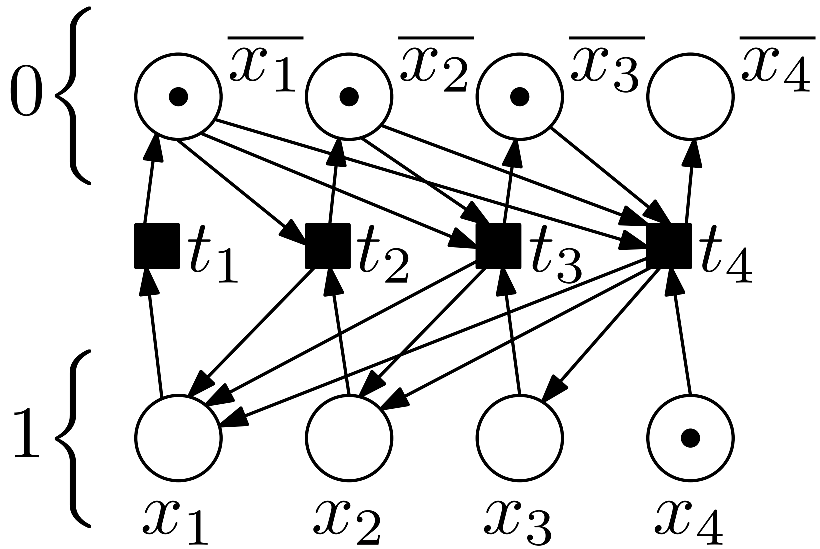

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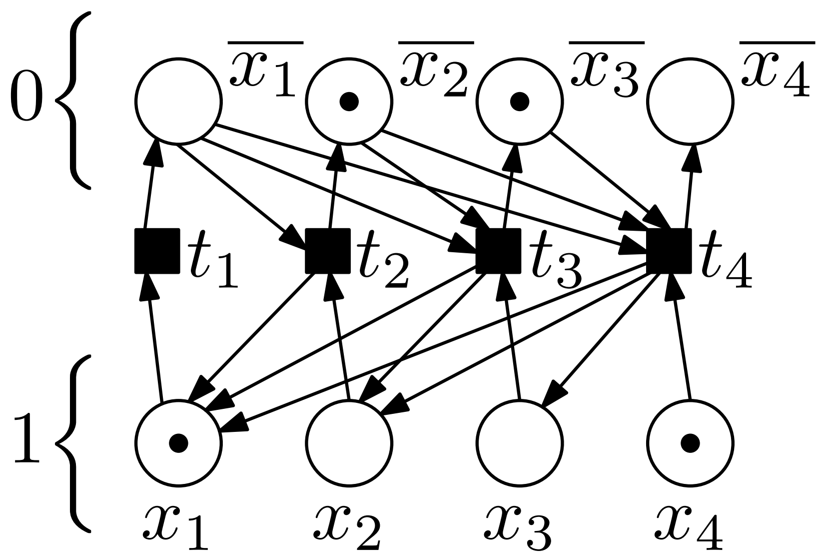

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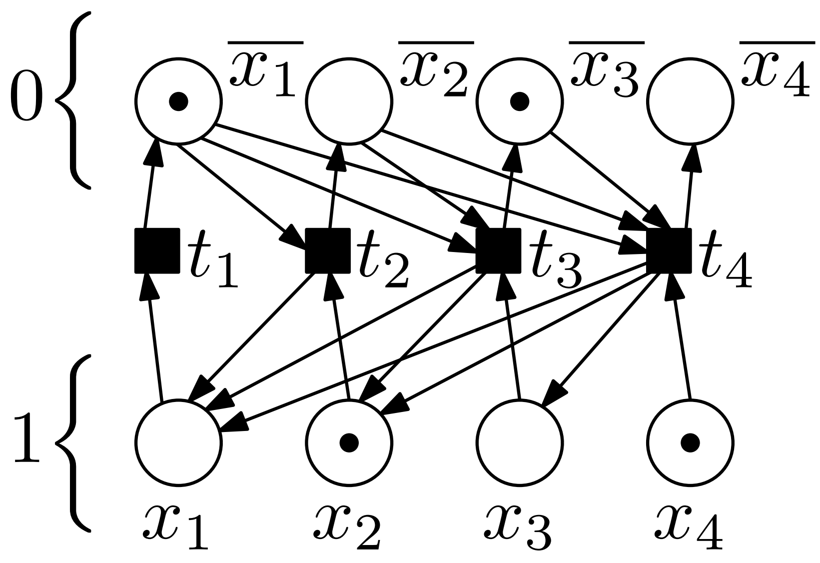

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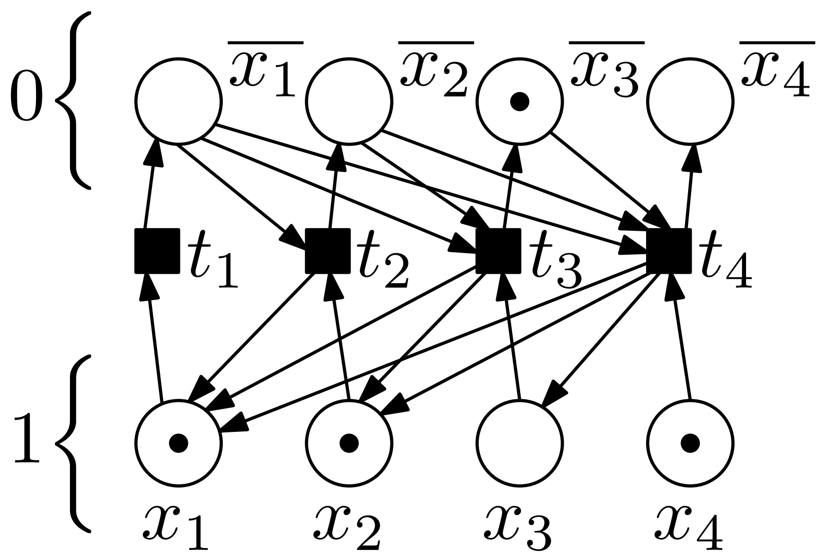

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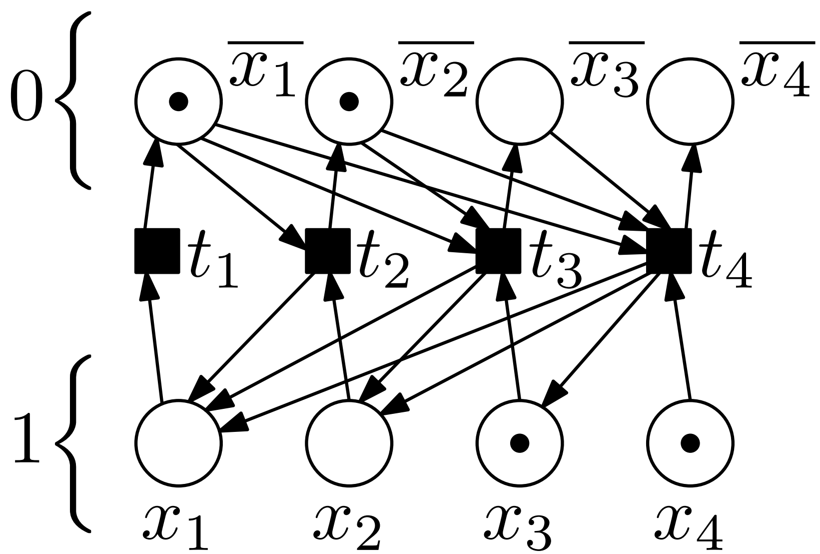

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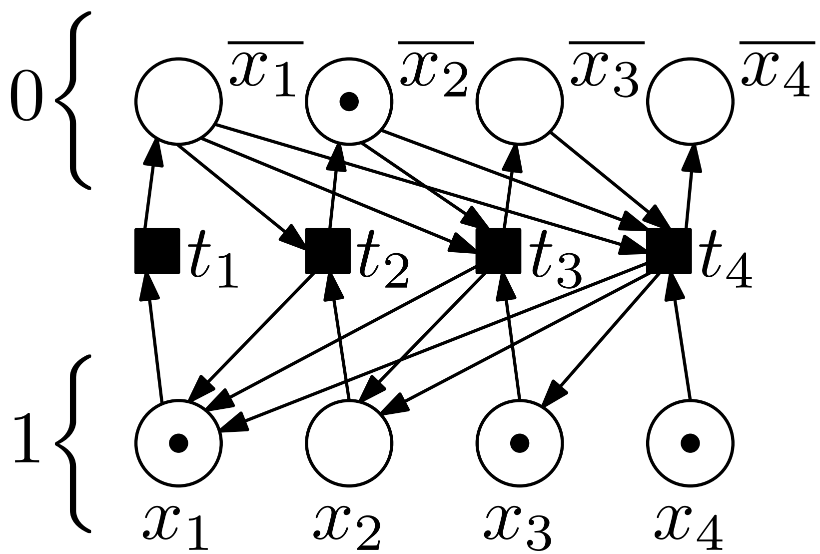

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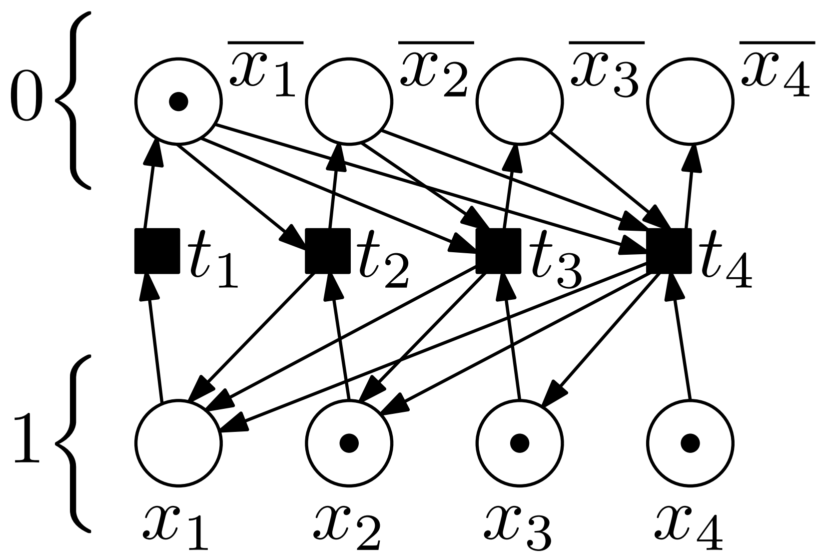

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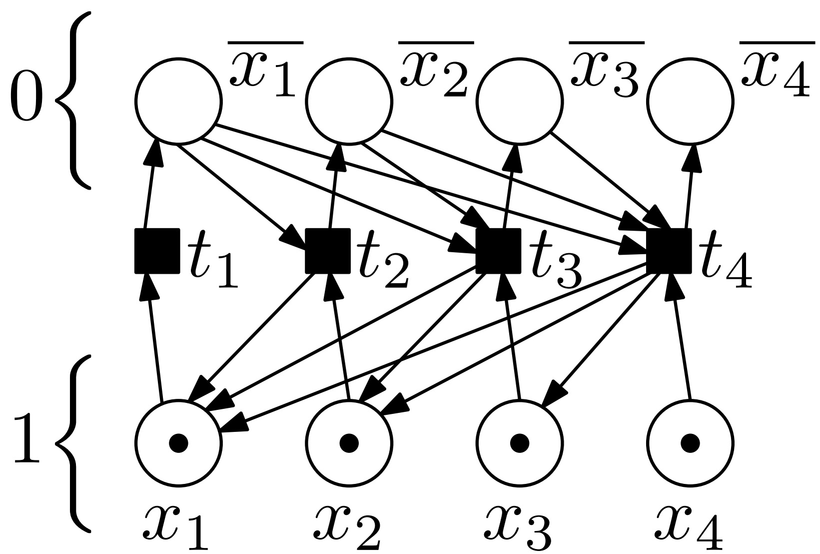

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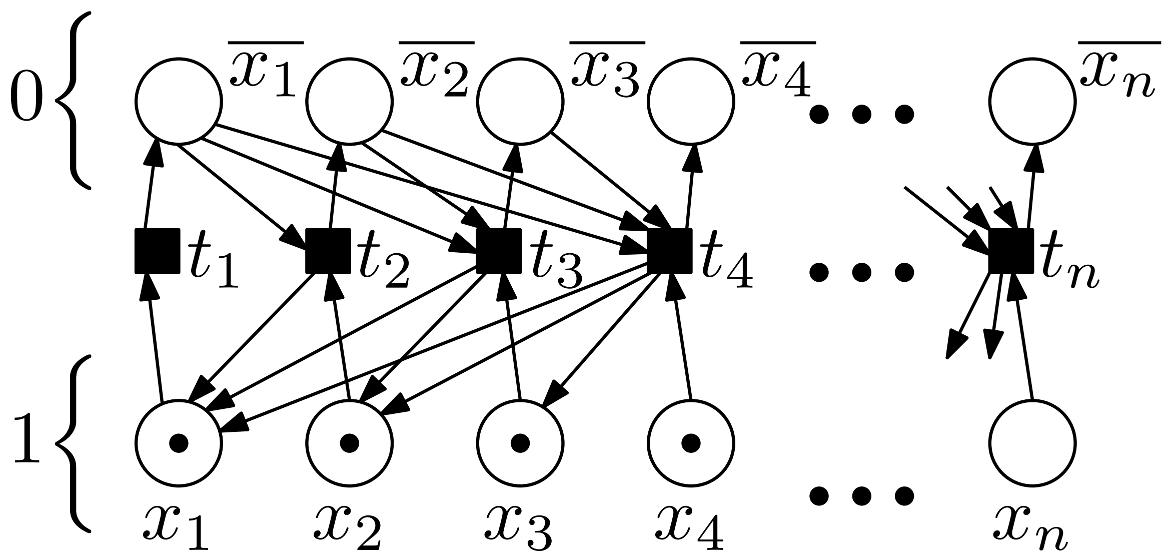

Proof ideas

We want 2^n

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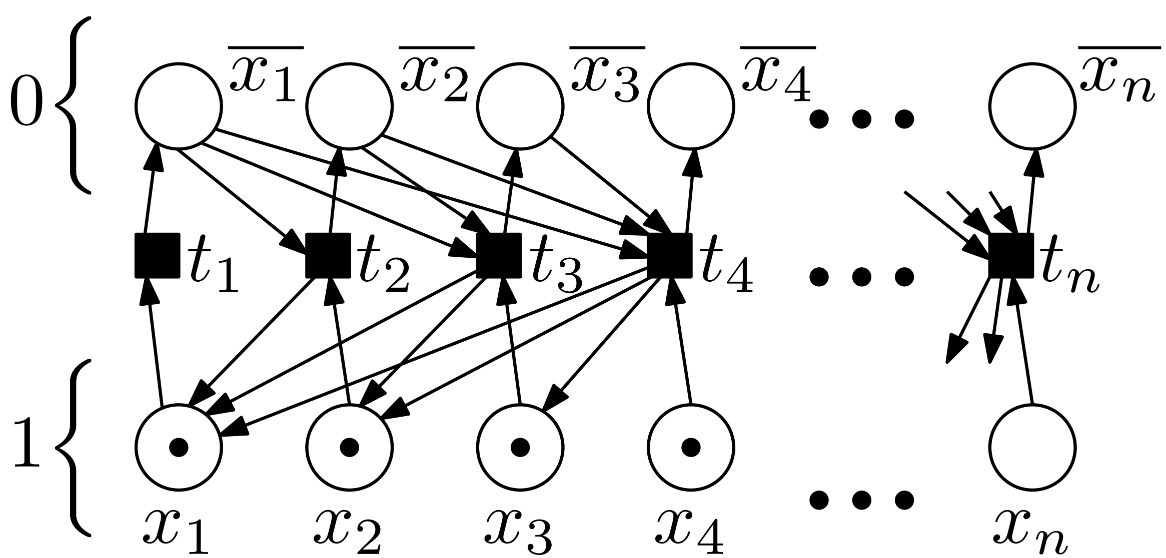

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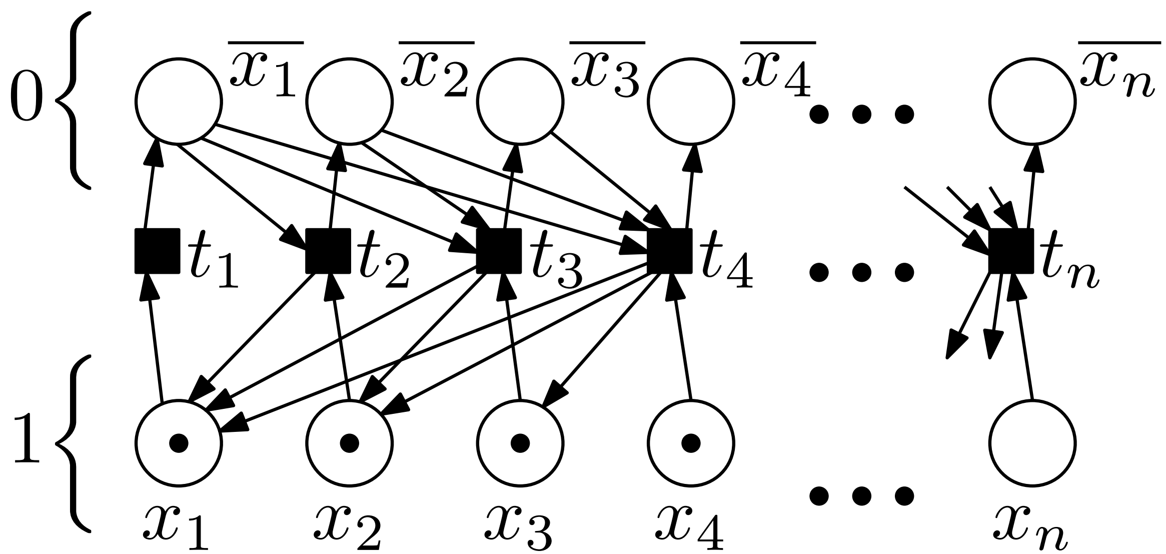

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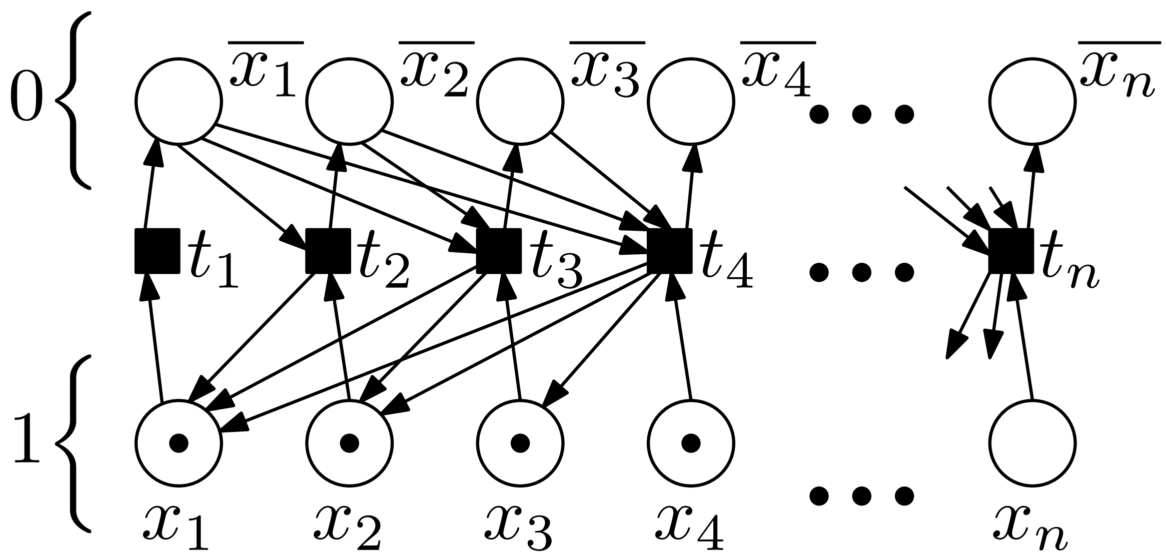
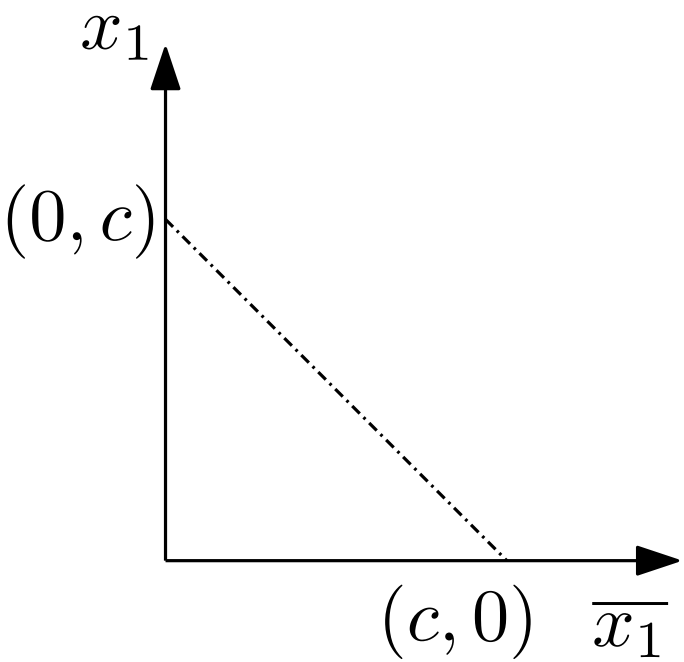
2^n

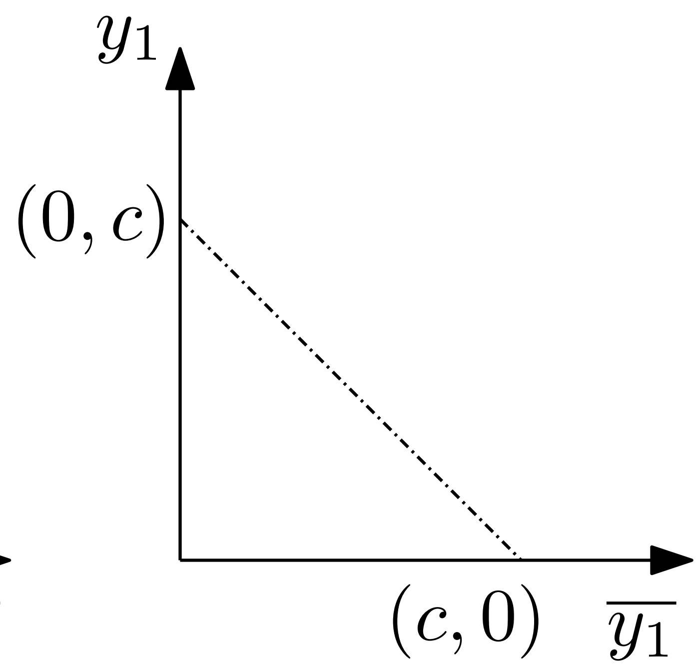
First idea, a

$x_i + \overline{x_i}$ constant for every i yields 2^n incomparable



Towards double exponential





Let $c = 2^{2^i}$, then $\{(x_1, \overline{x_1}, y_1, \overline{y_1}) \mid x_1 + \overline{x_1} = c,\ y_1 + \overline{y_1} = c\}$ has $c \cdot c = 2^{2^{i+1}}$ incomparable states

Recall that $\Delta_i \subseteq Q_i \times [-1, 1]^d \times Q_i$, so direct increment / decrement / test of 2^{2^i} is not allowed.
Moreover, setting initial value to 2^{2^i} is not allowed.

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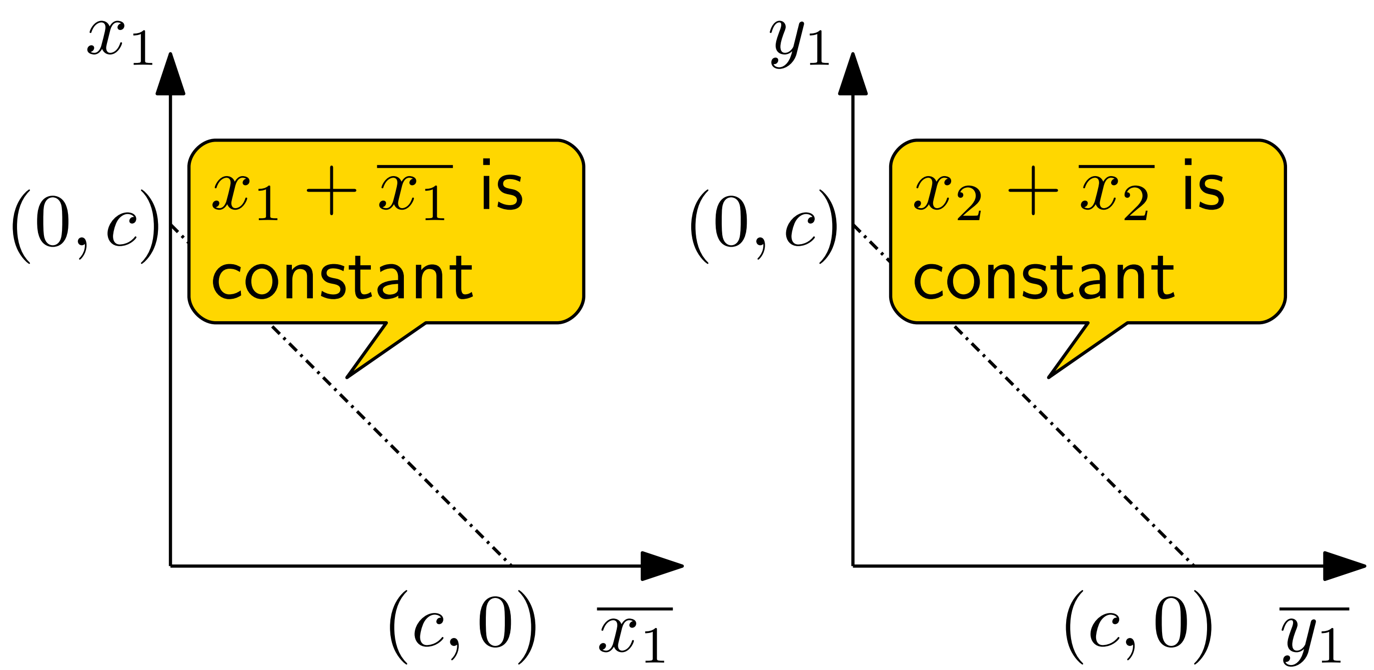

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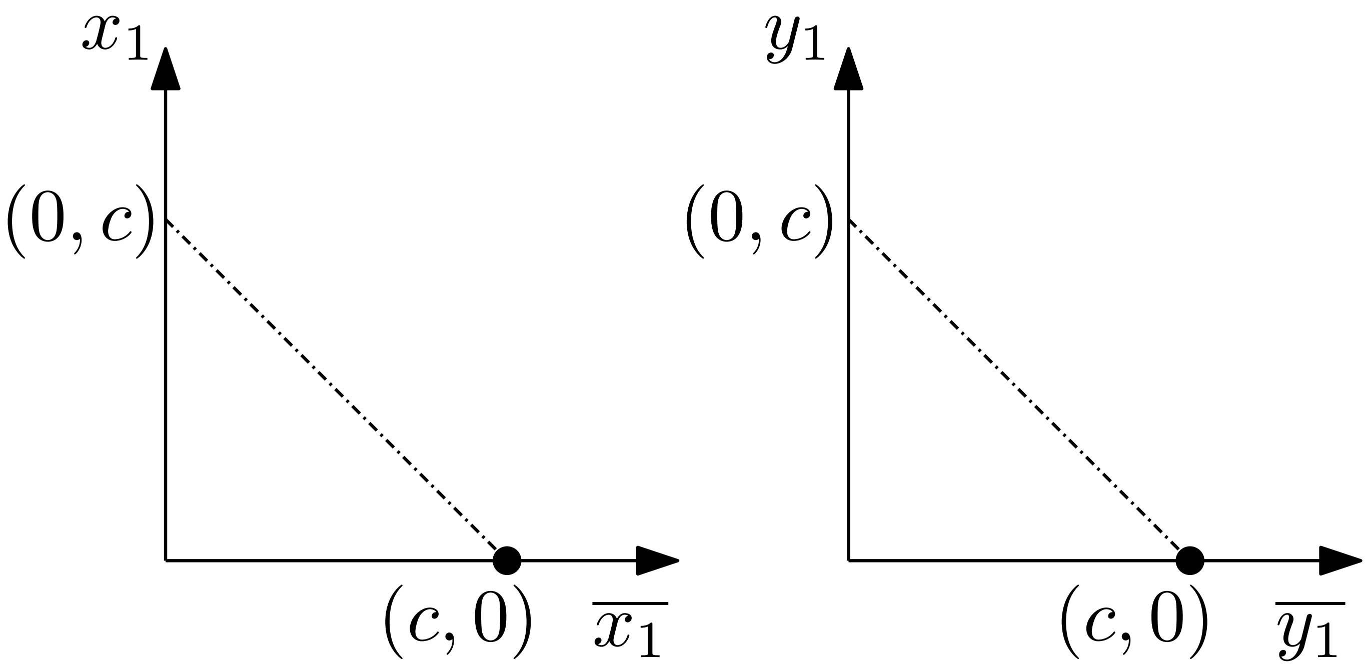

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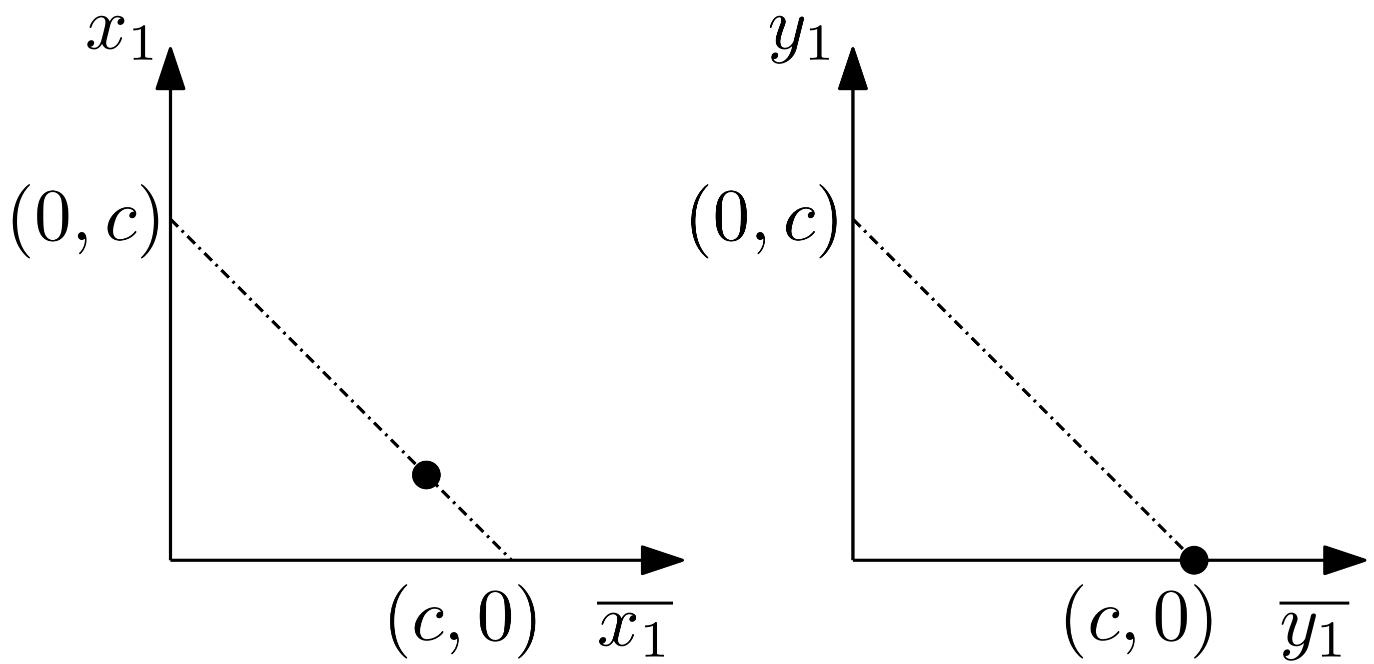

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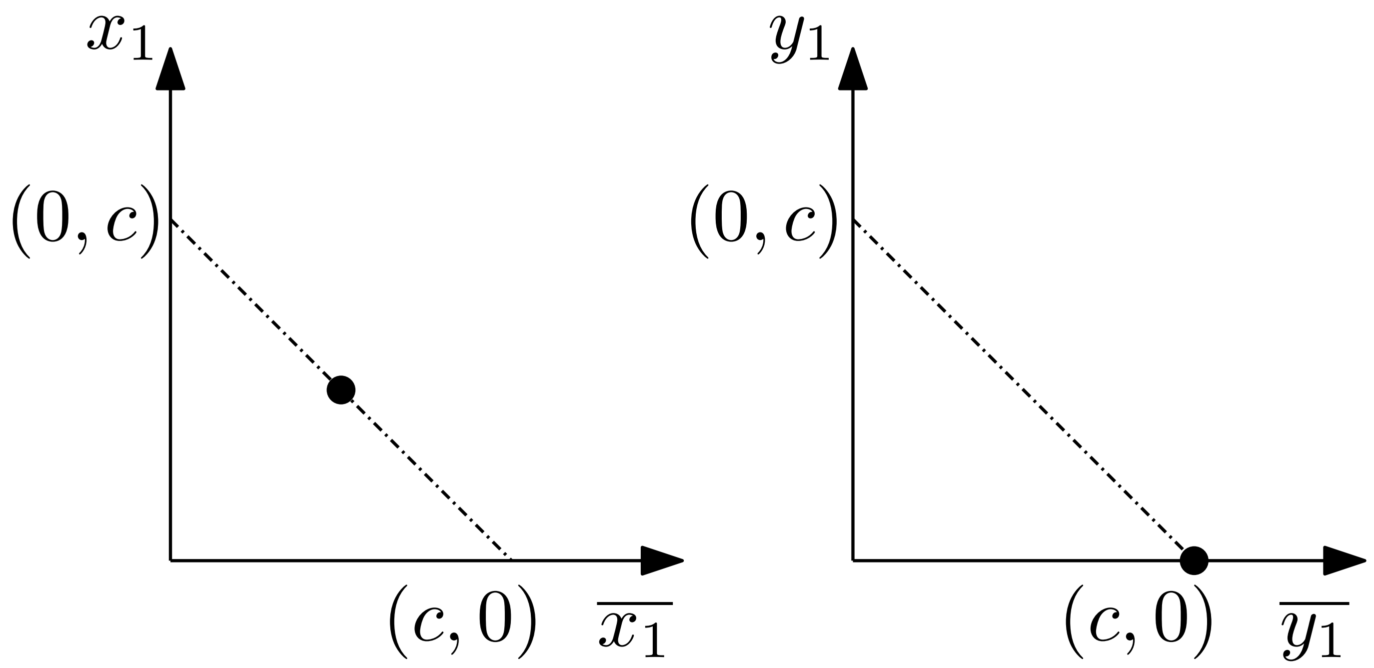

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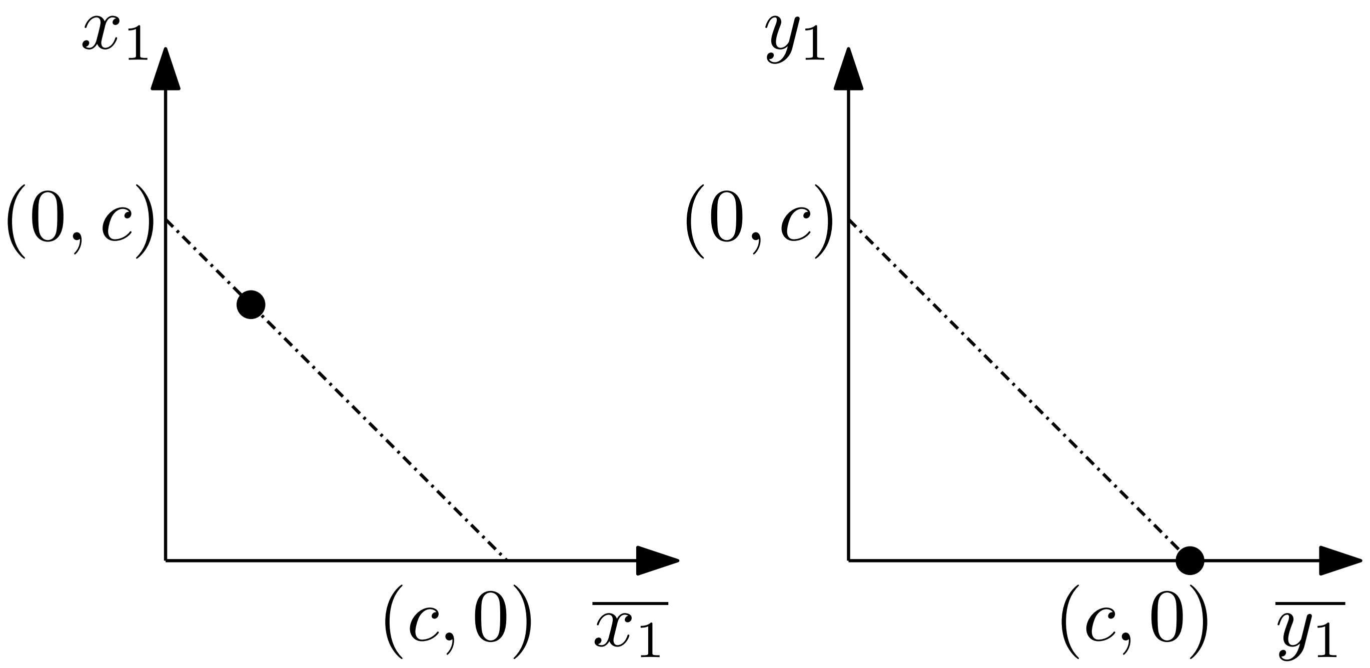

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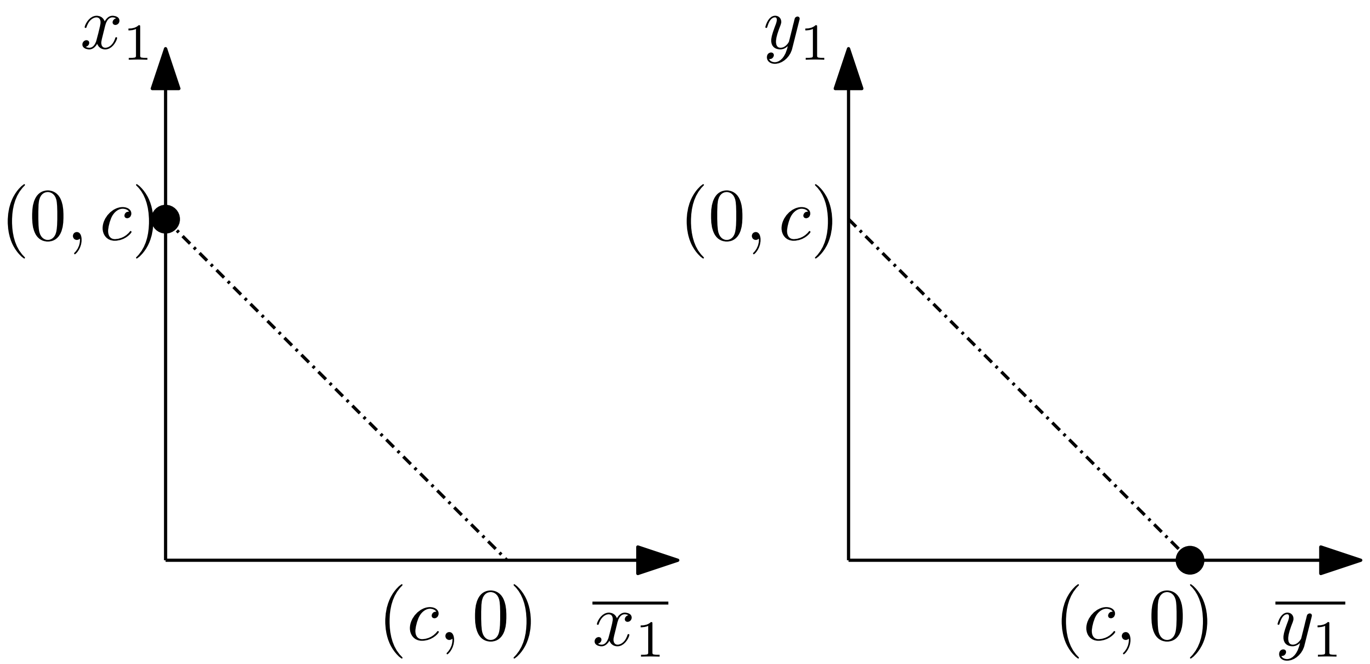

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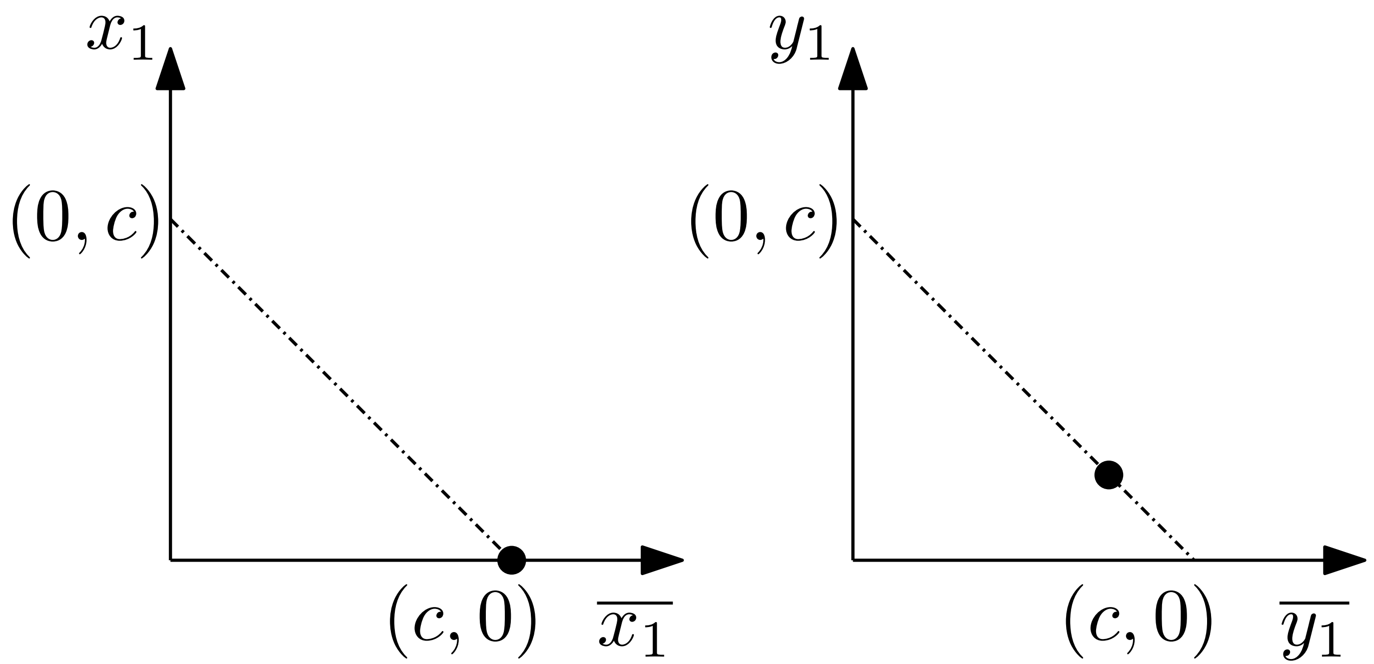

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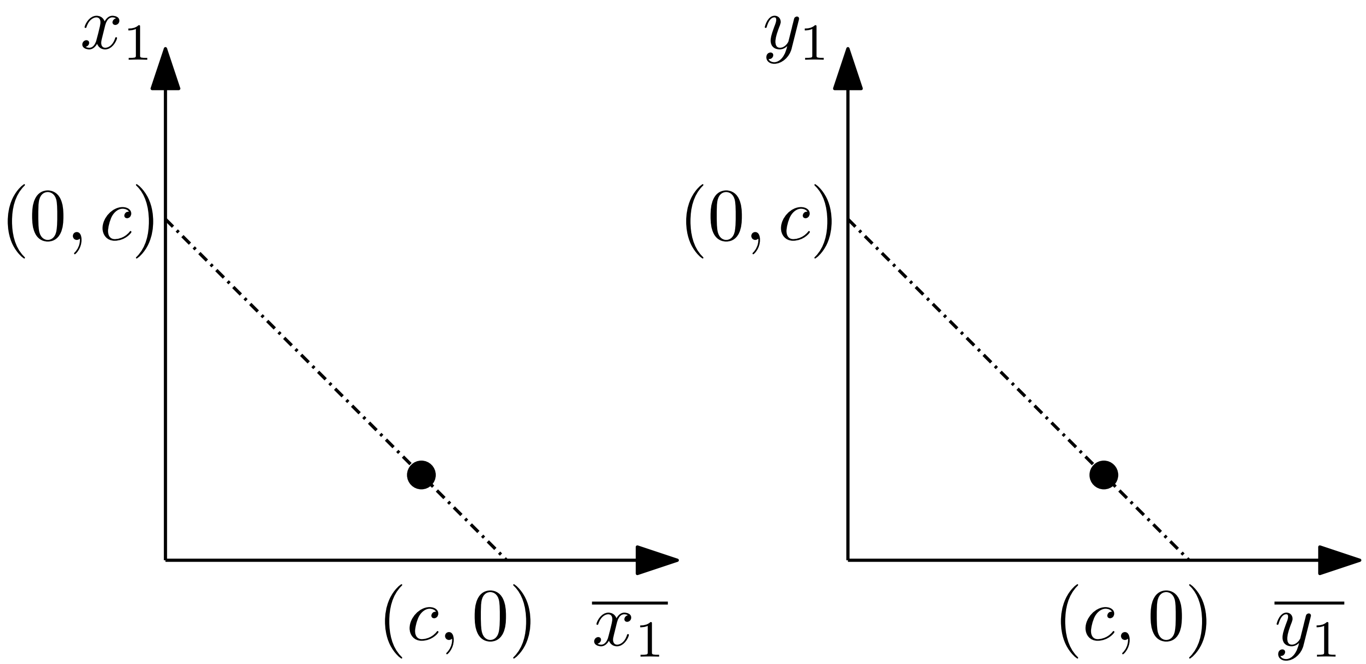

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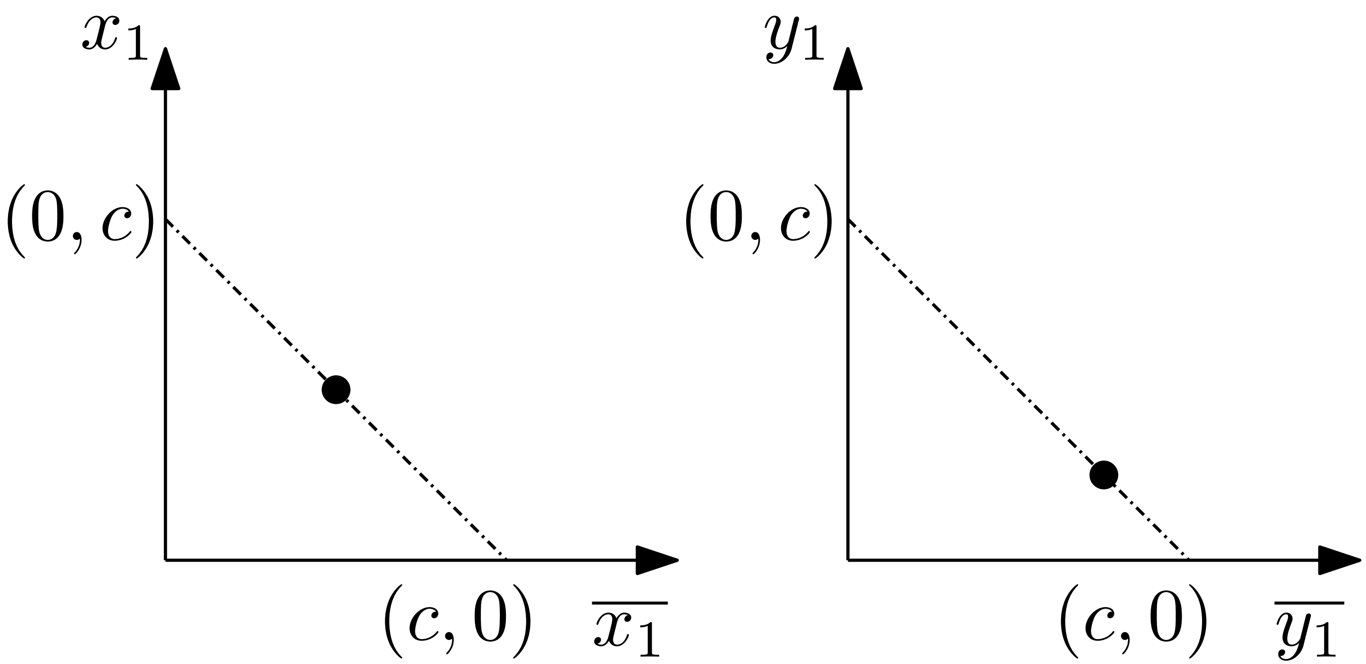
Moreover, setting initial value to 2^{2^i} is not allowed.

Towards double exponential



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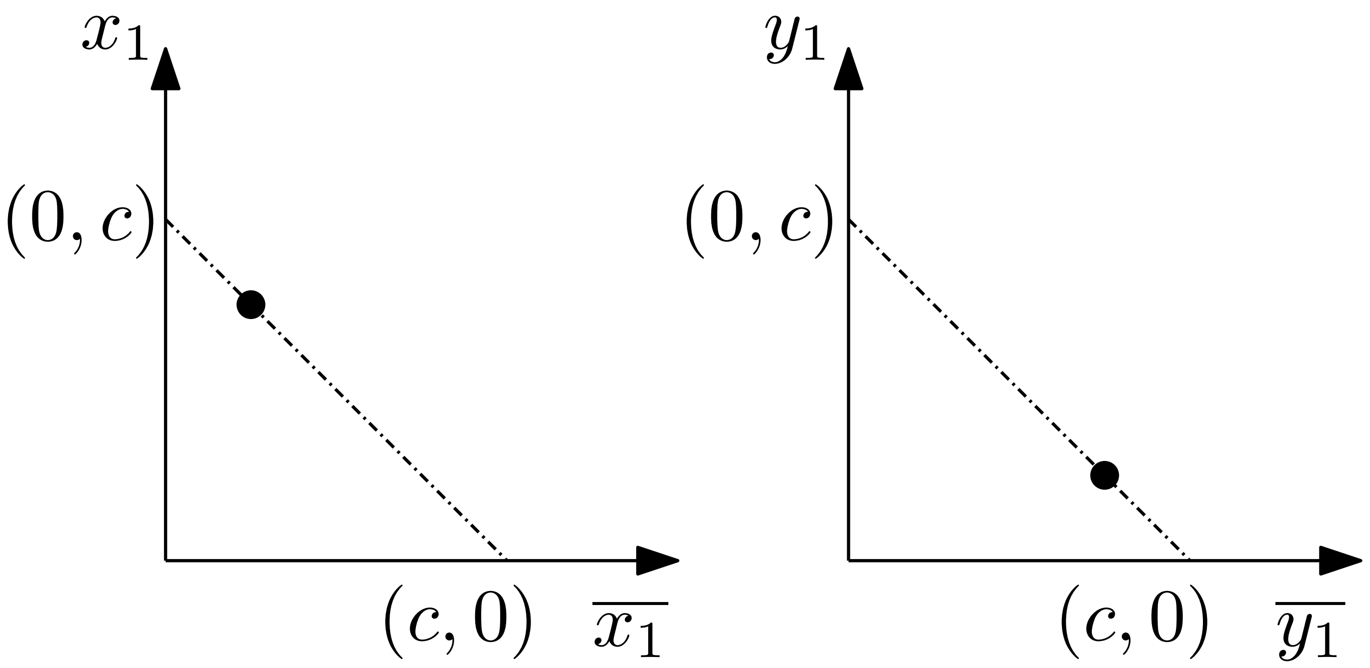

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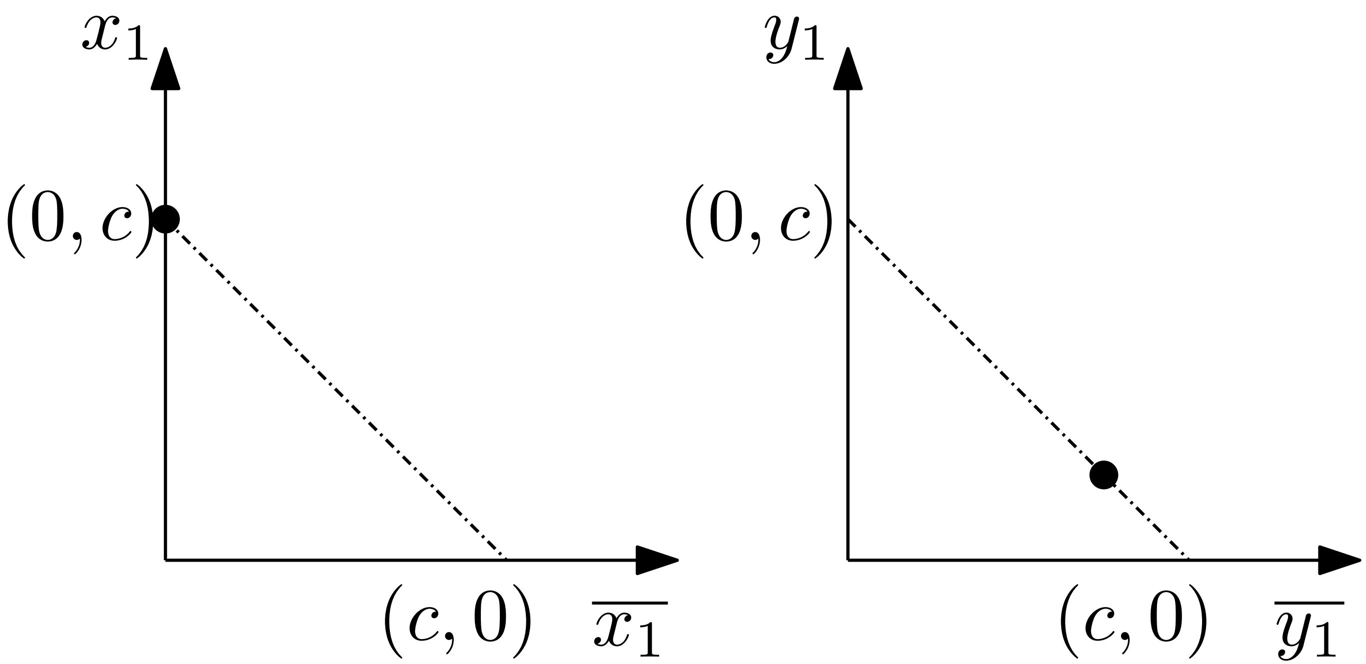

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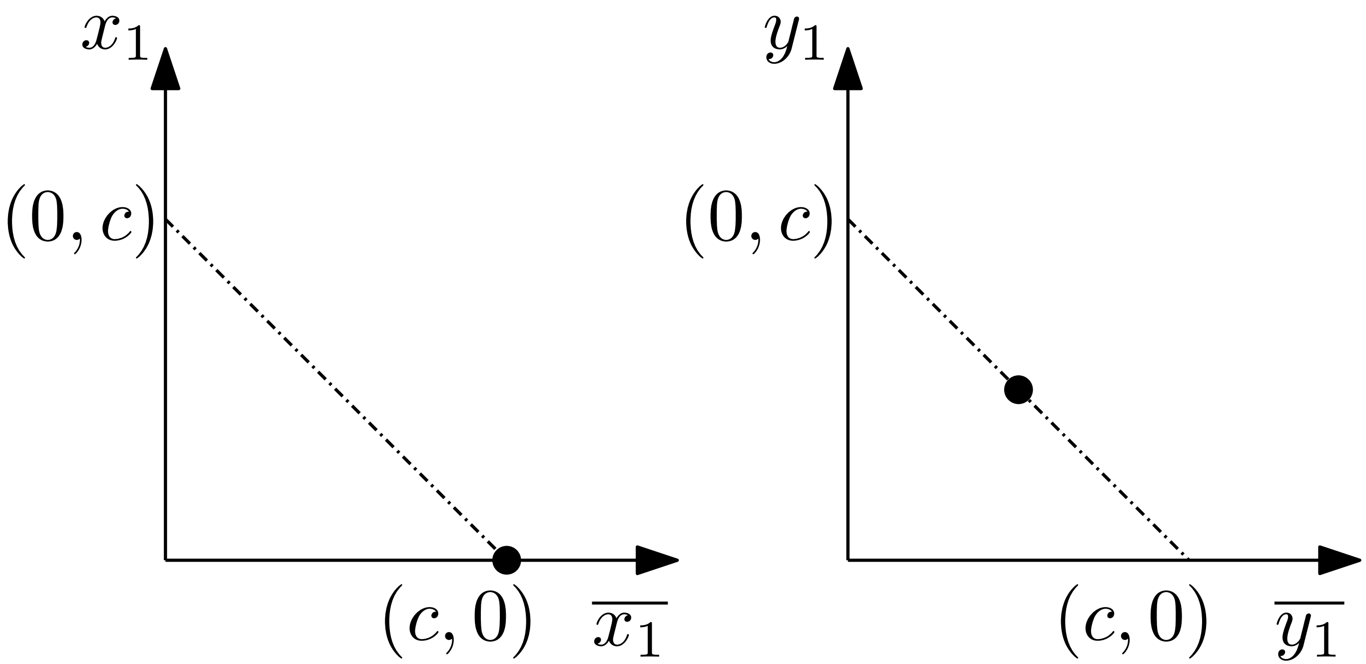

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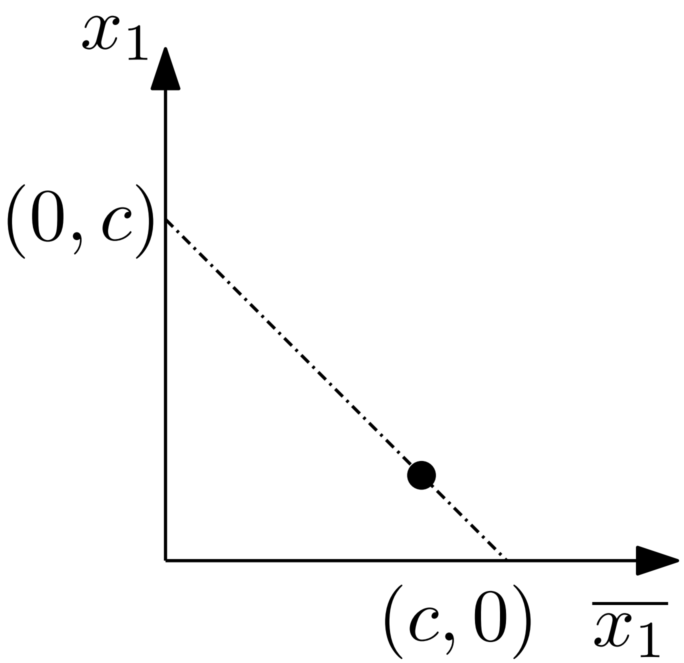

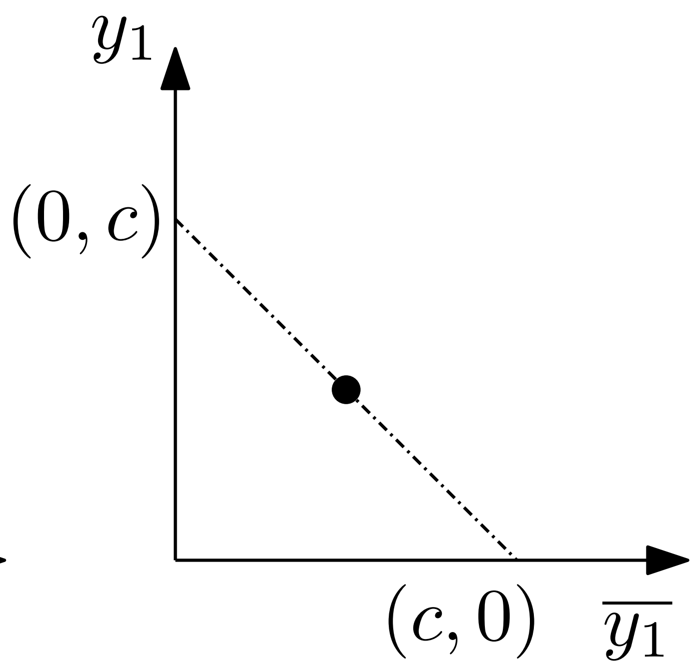

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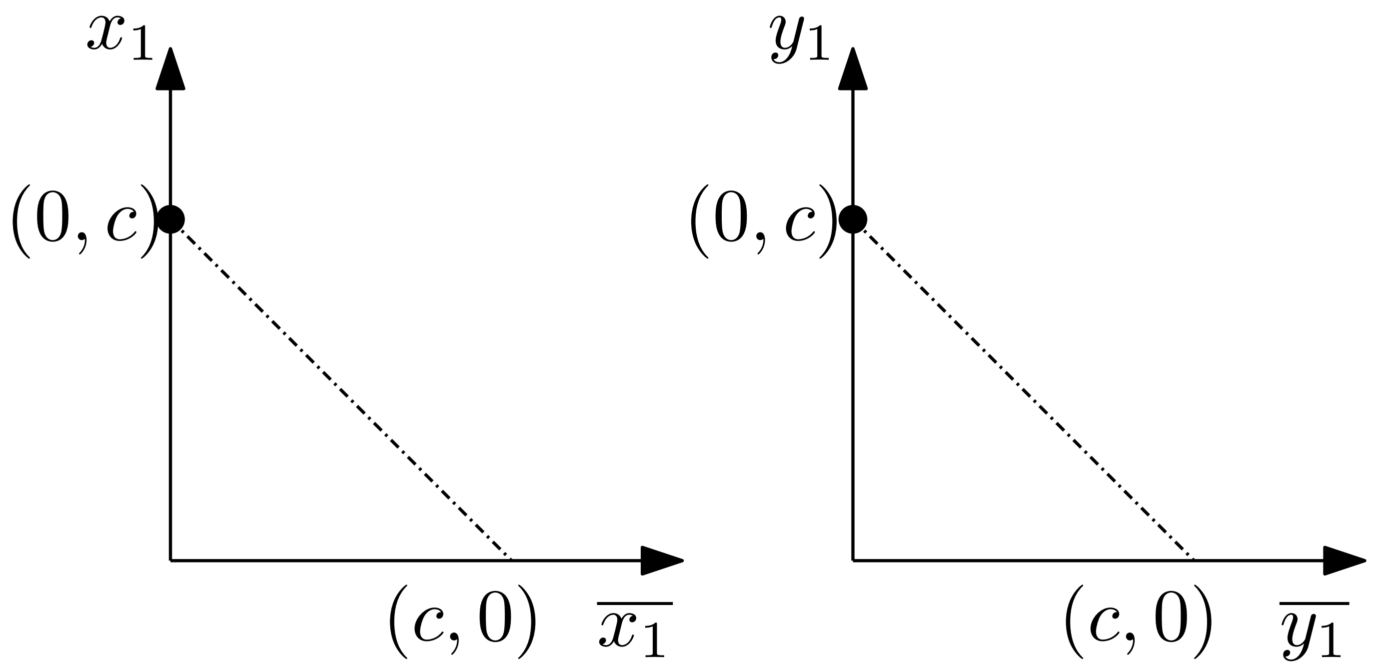

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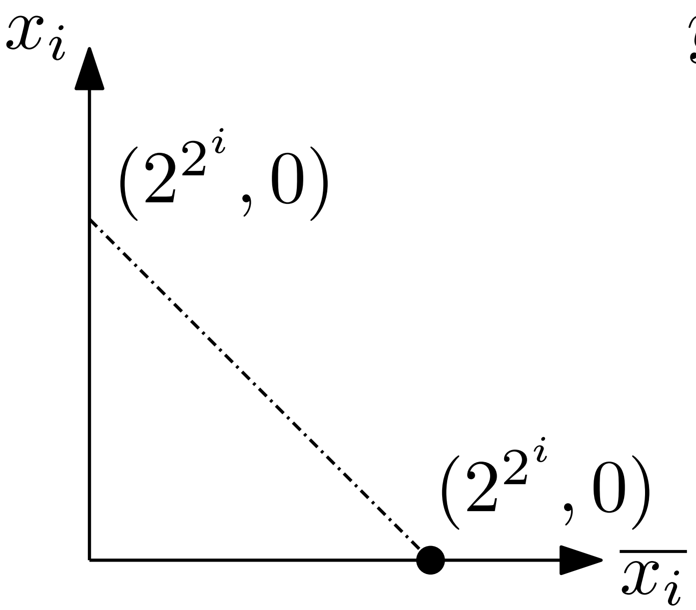

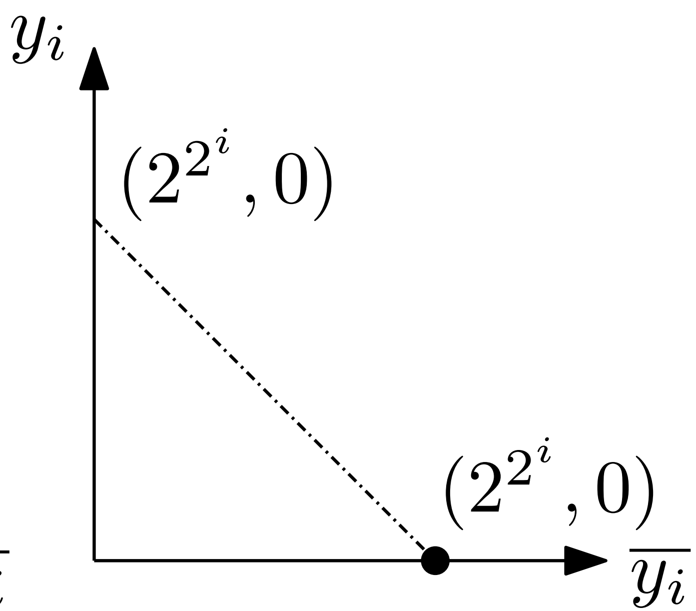

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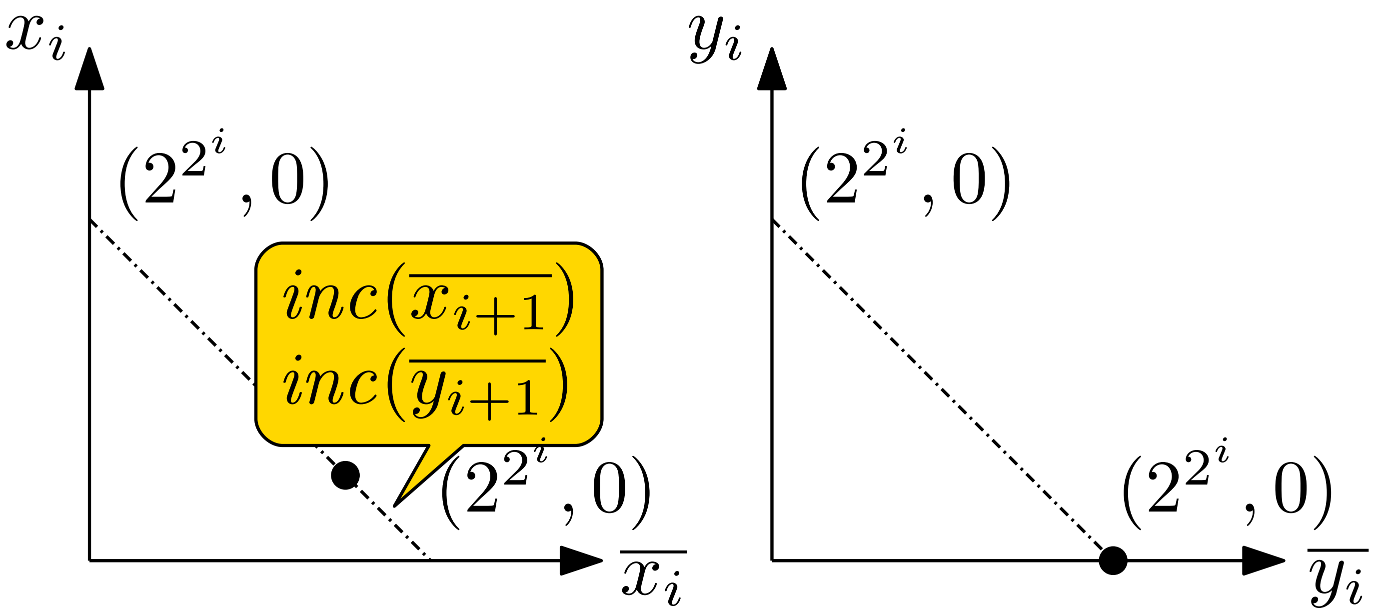

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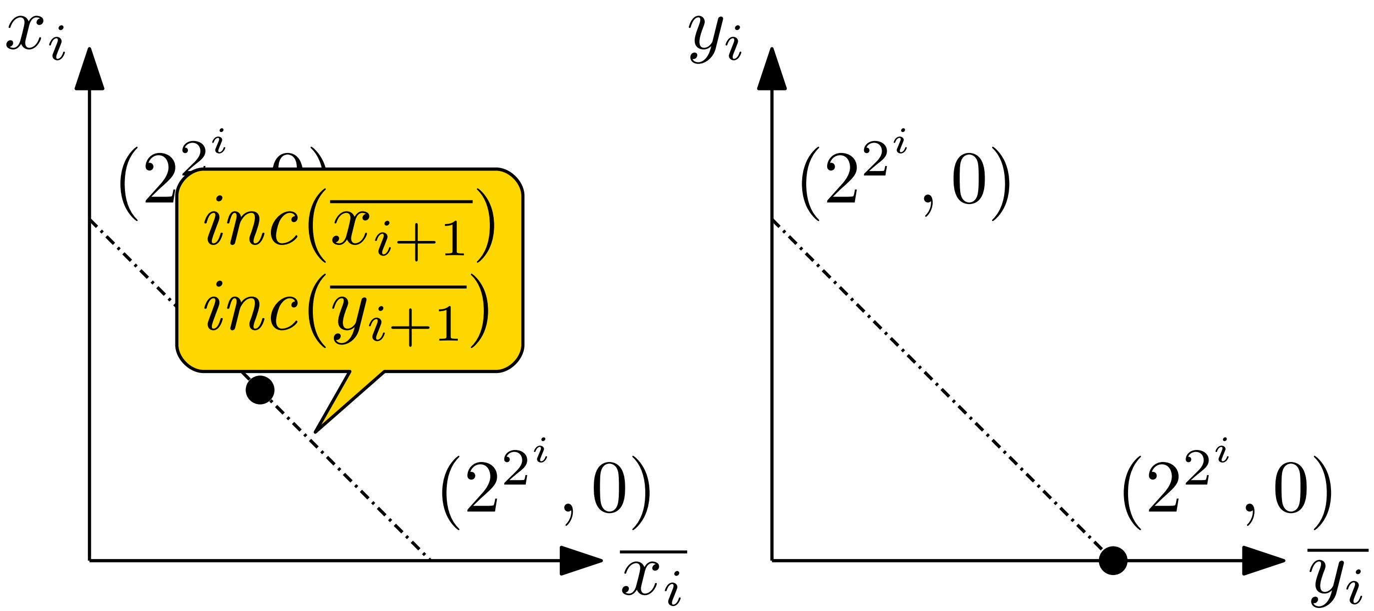

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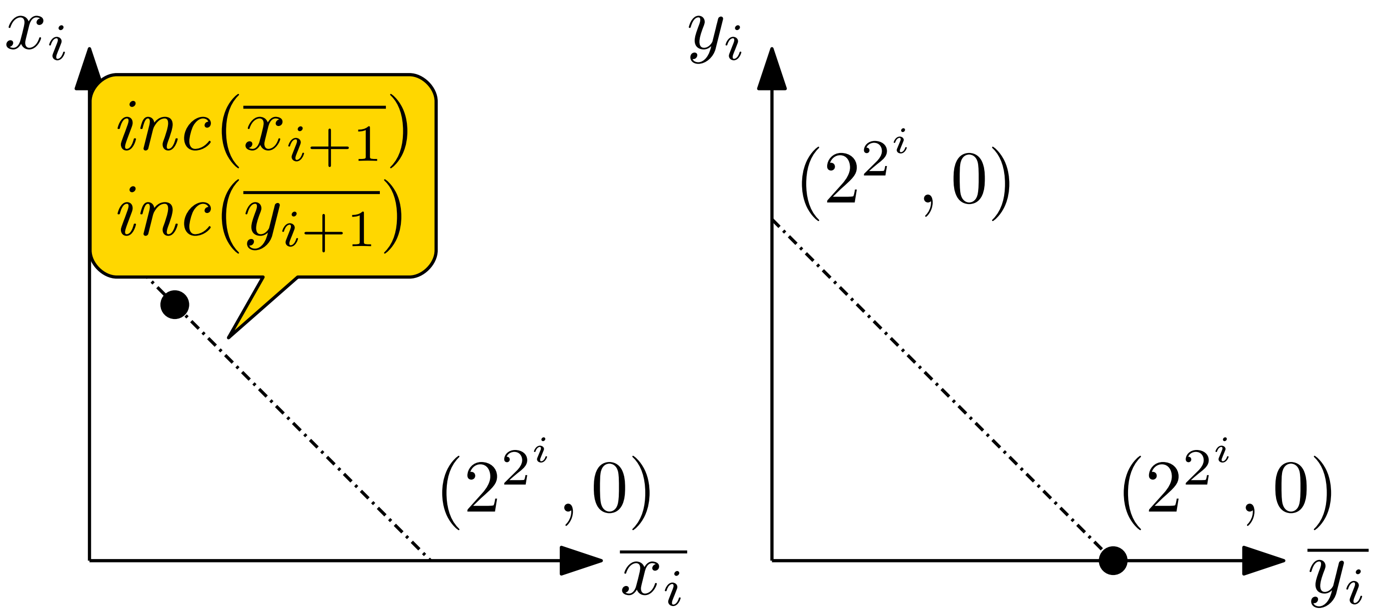

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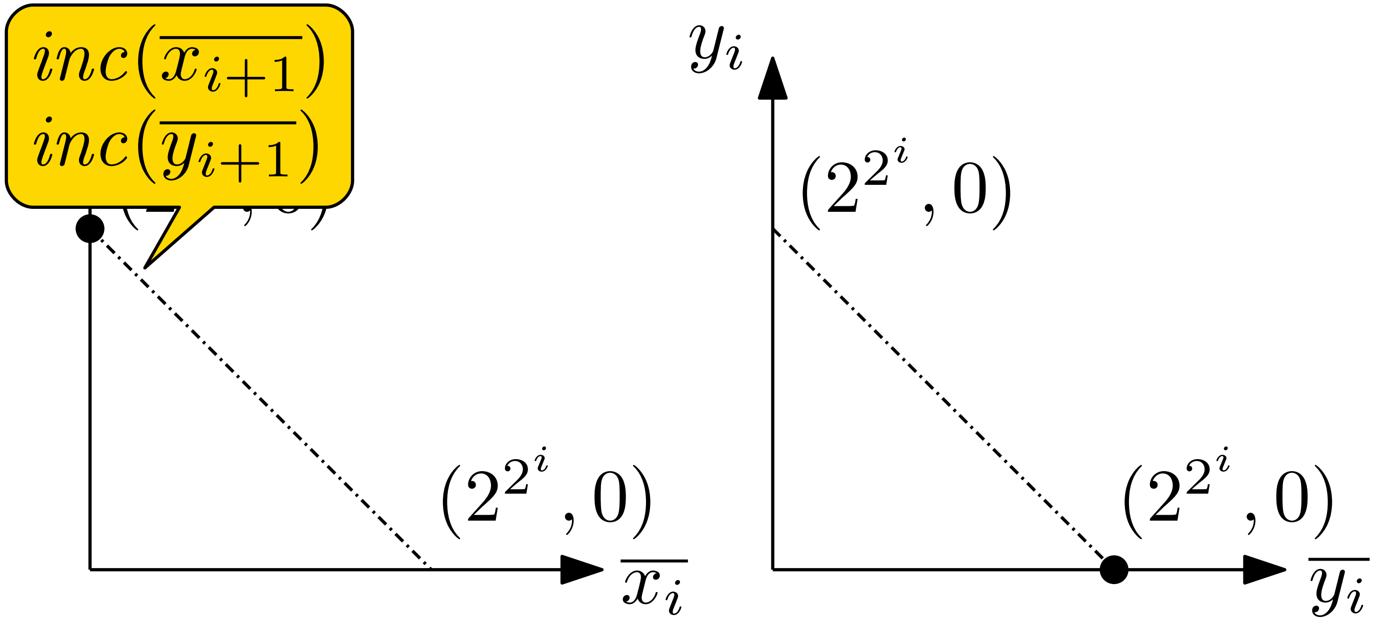

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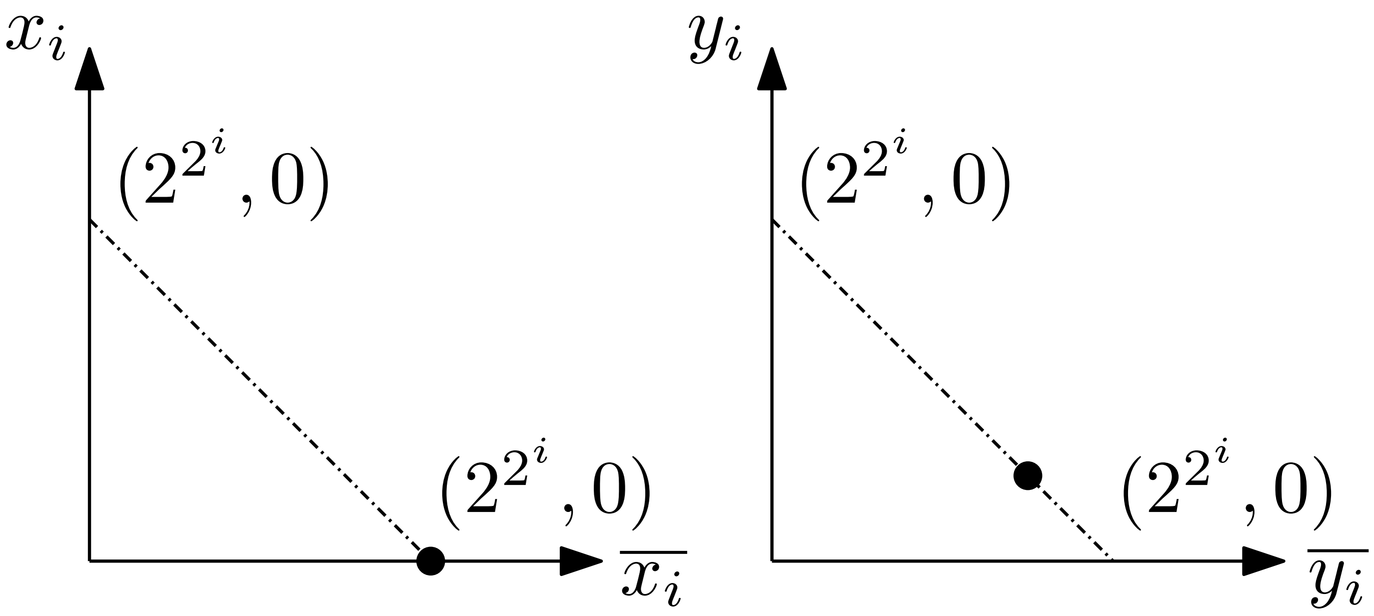

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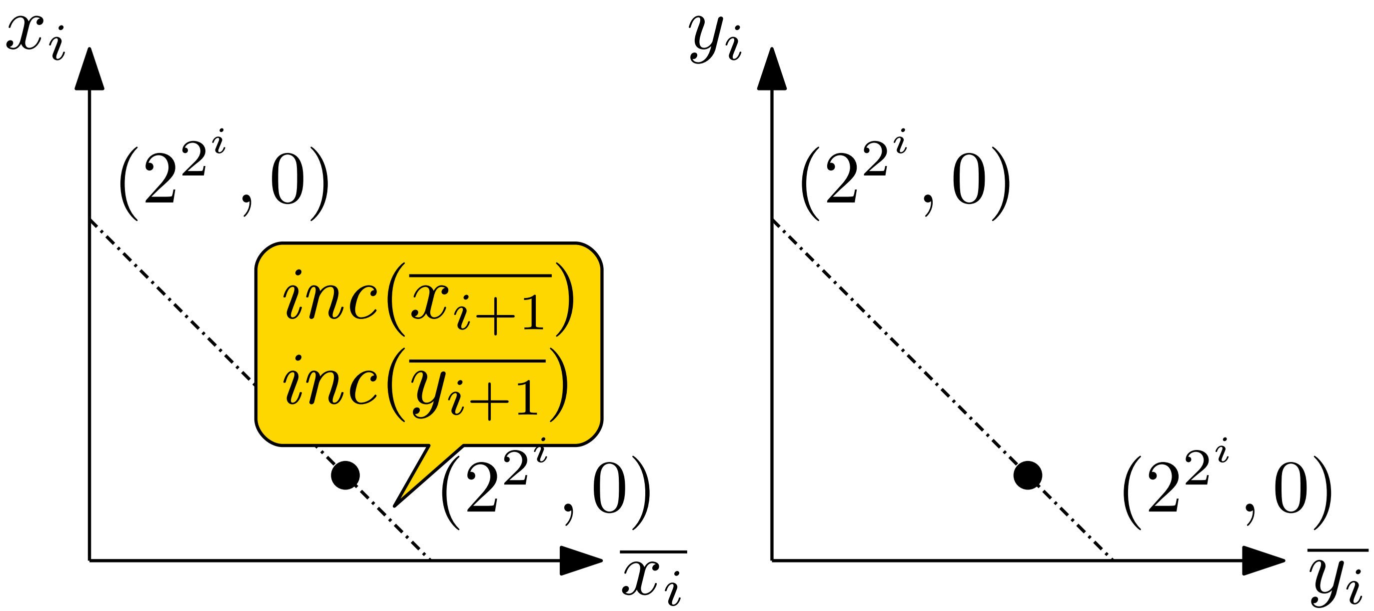

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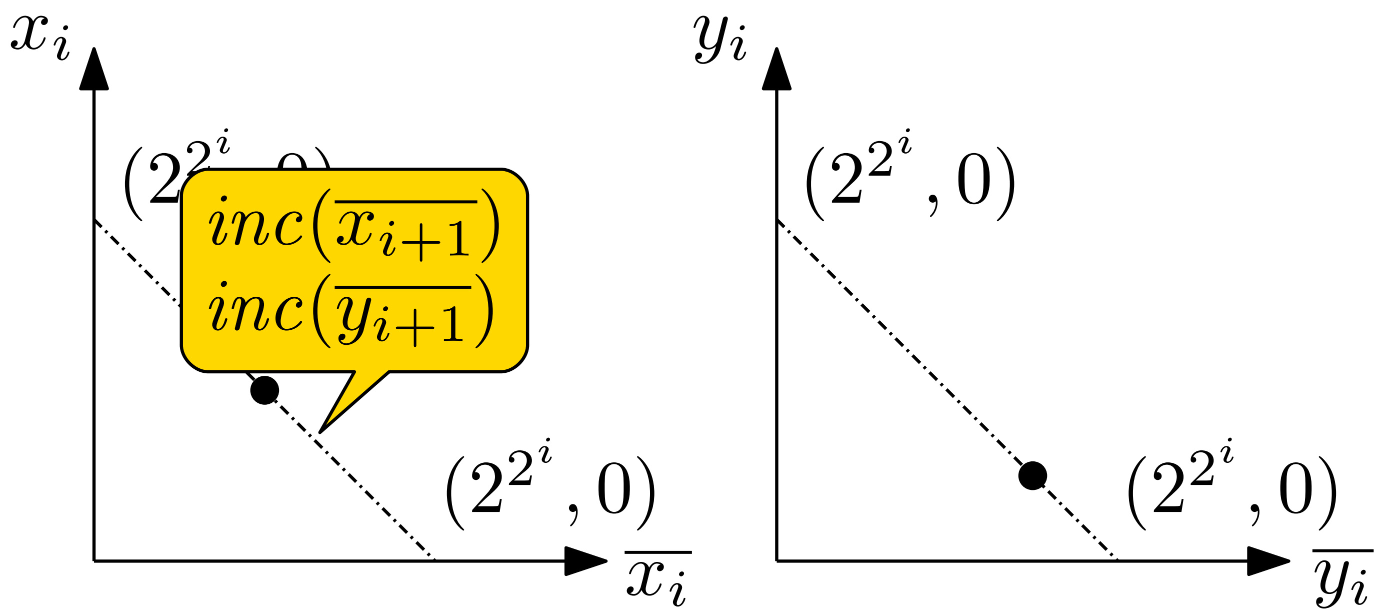

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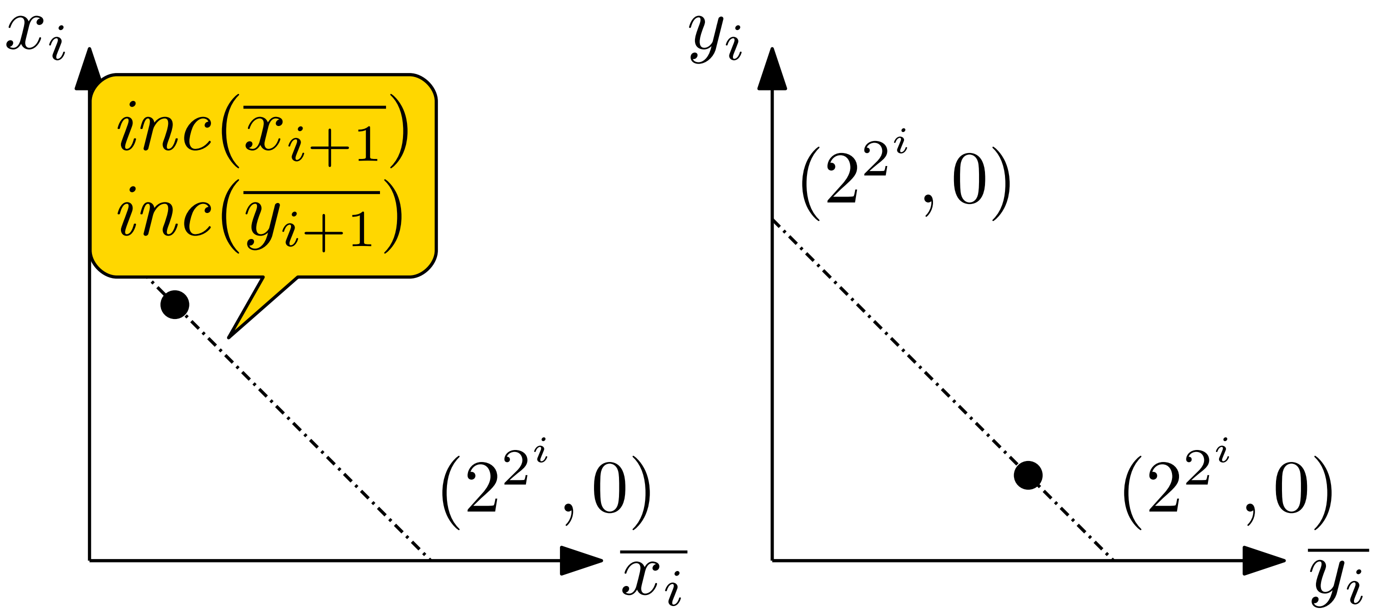

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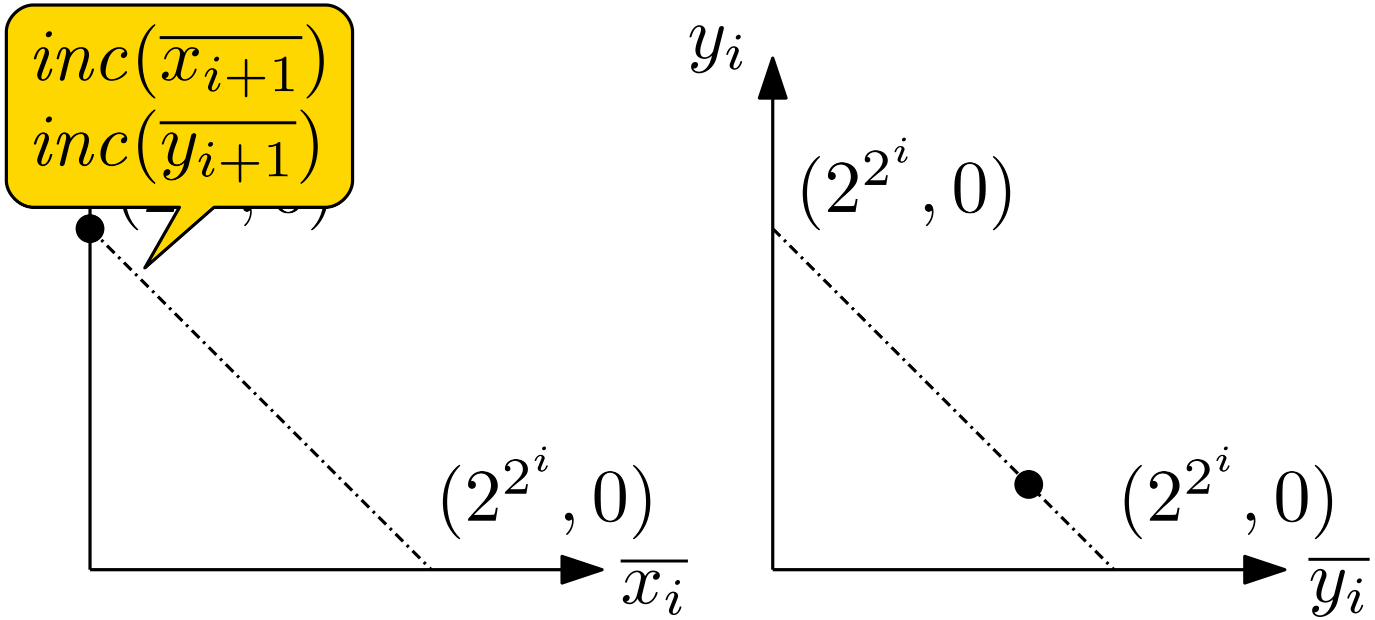

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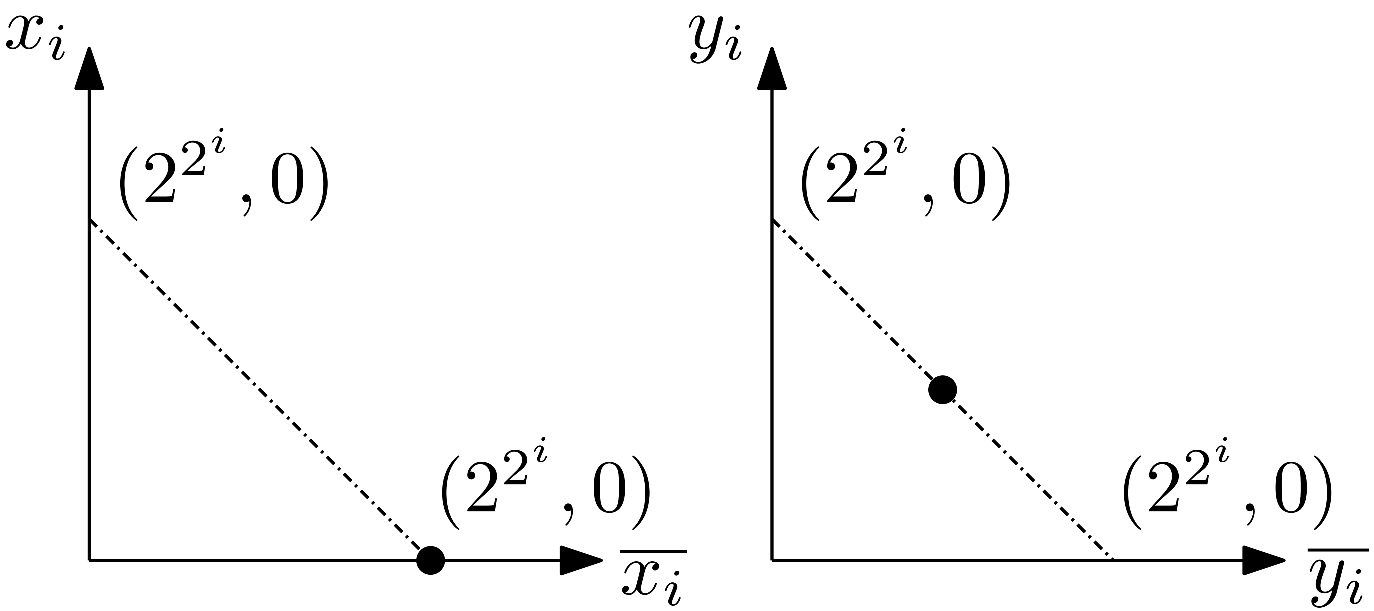

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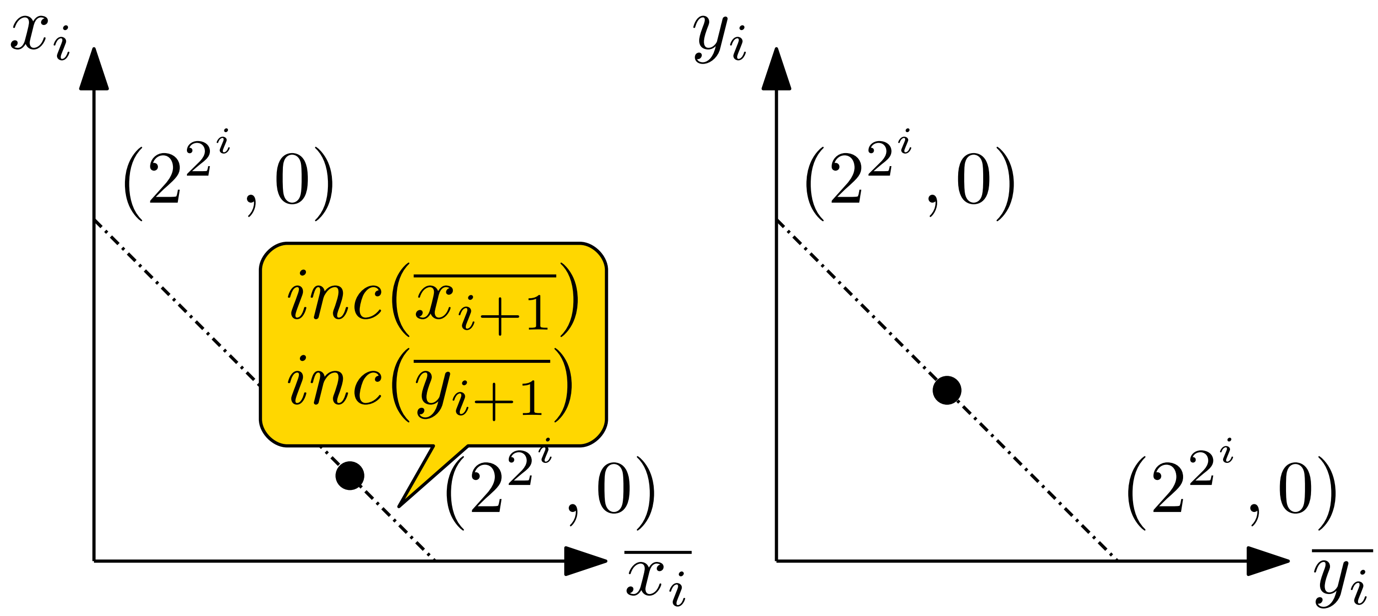

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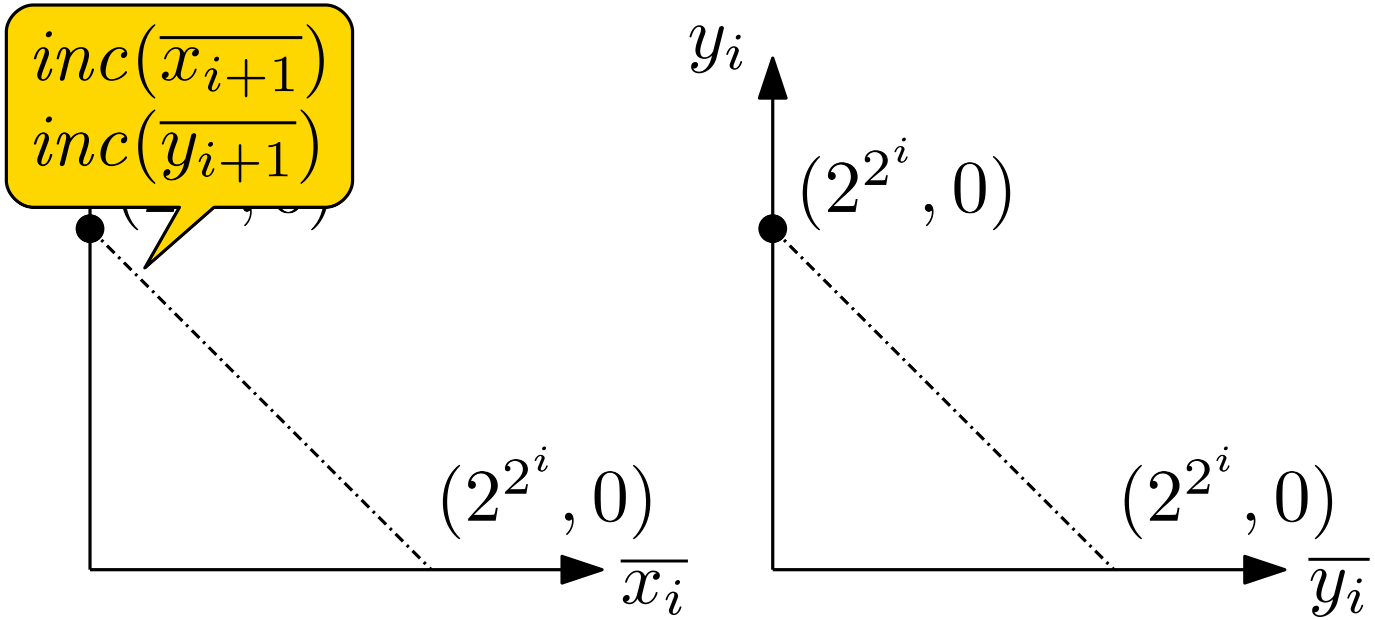

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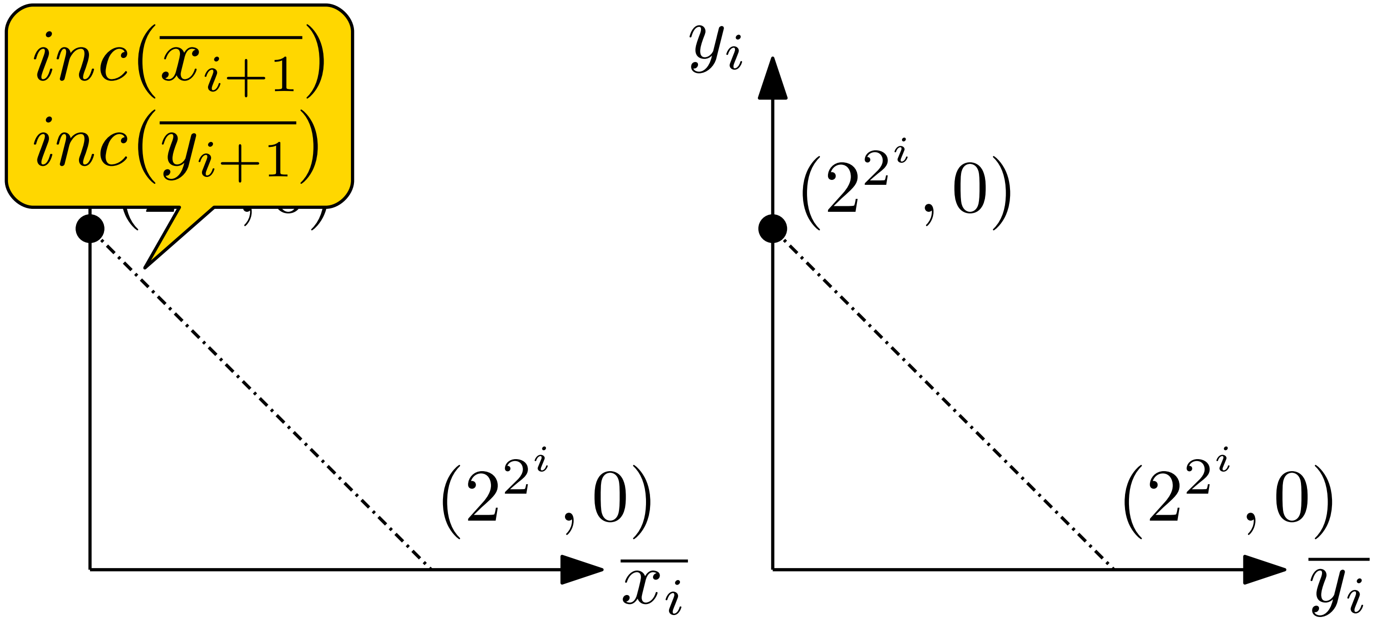

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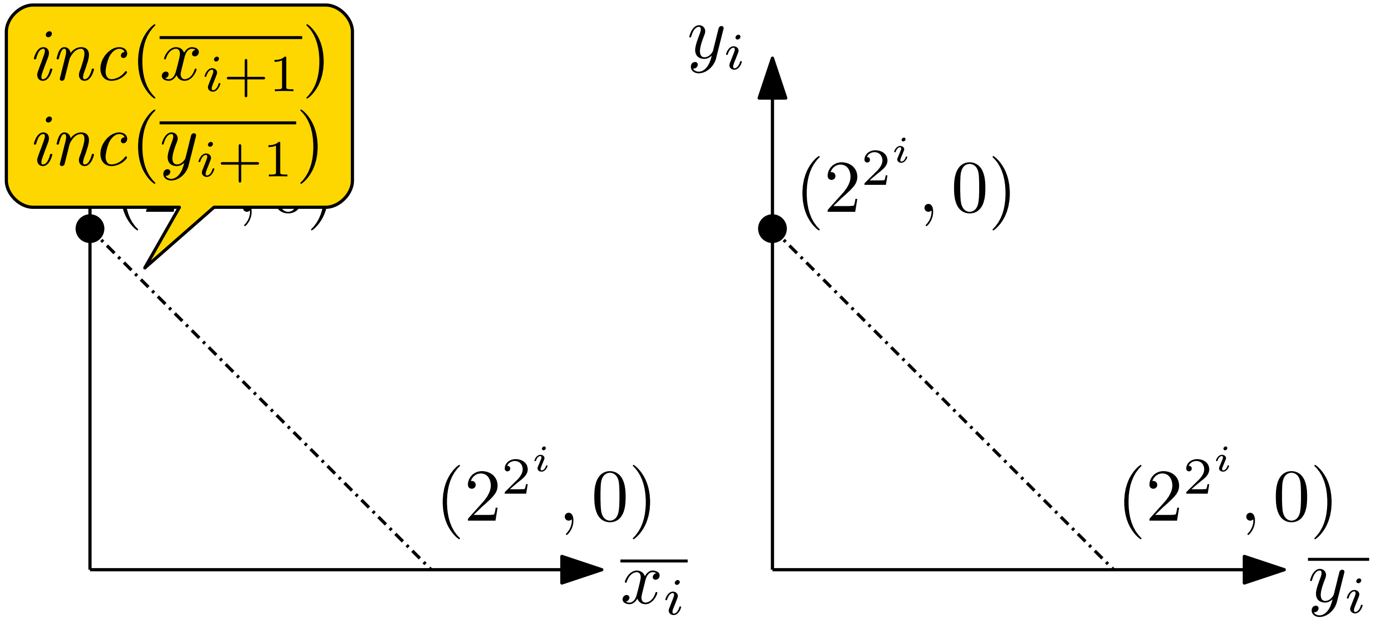

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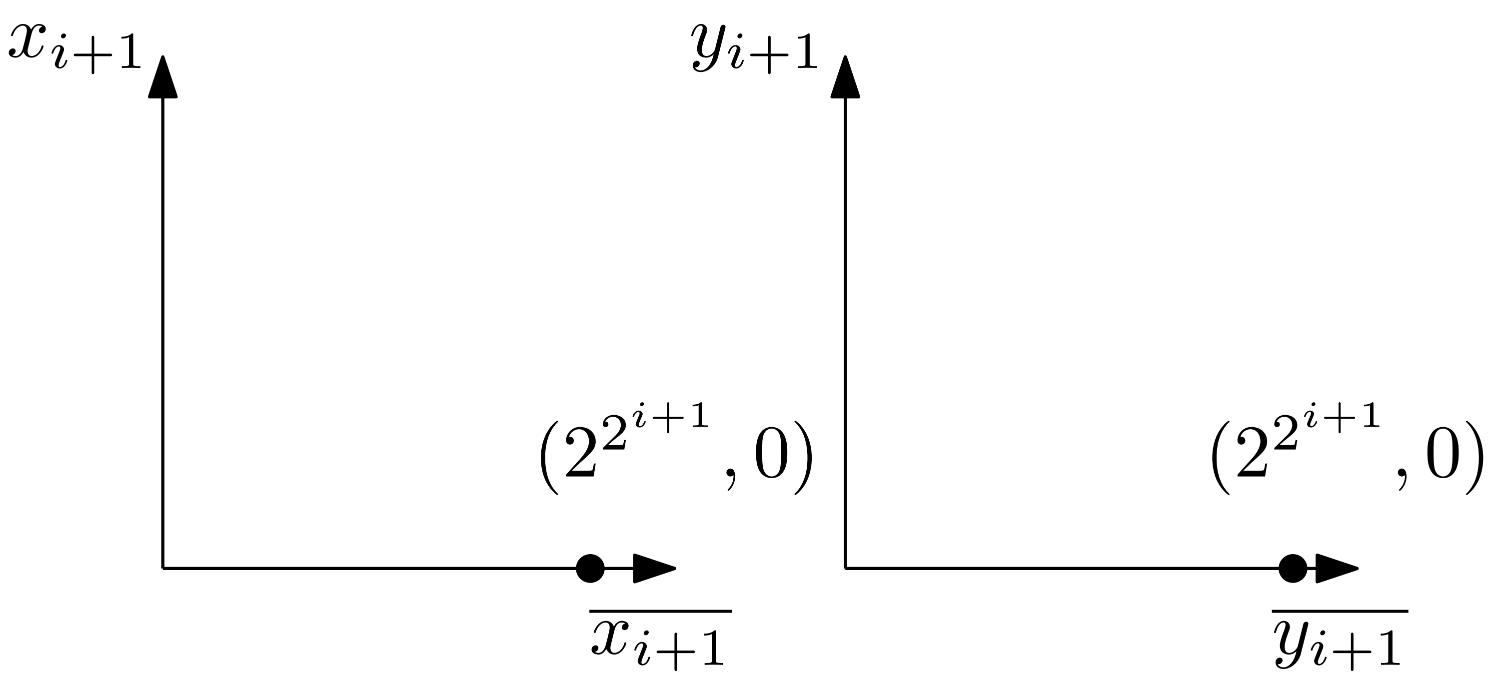

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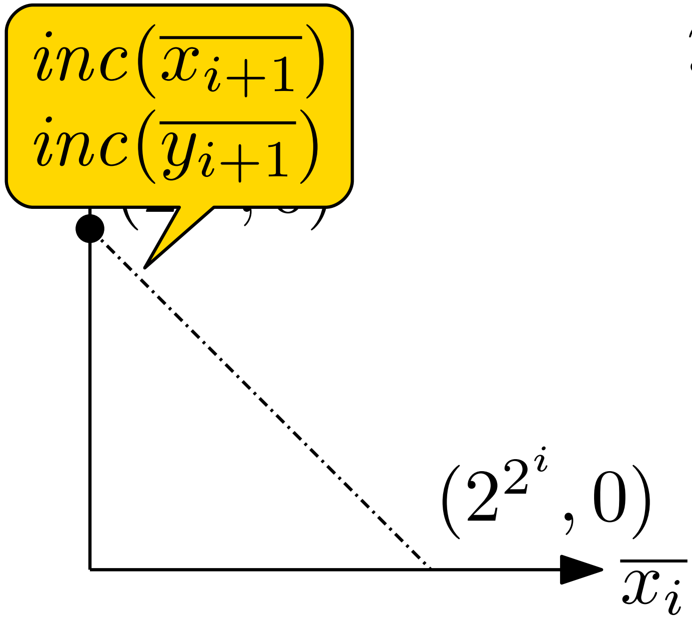

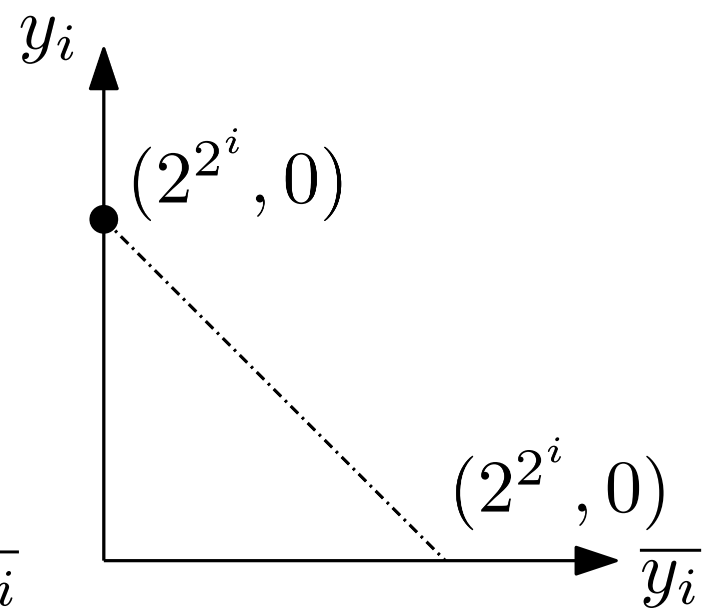

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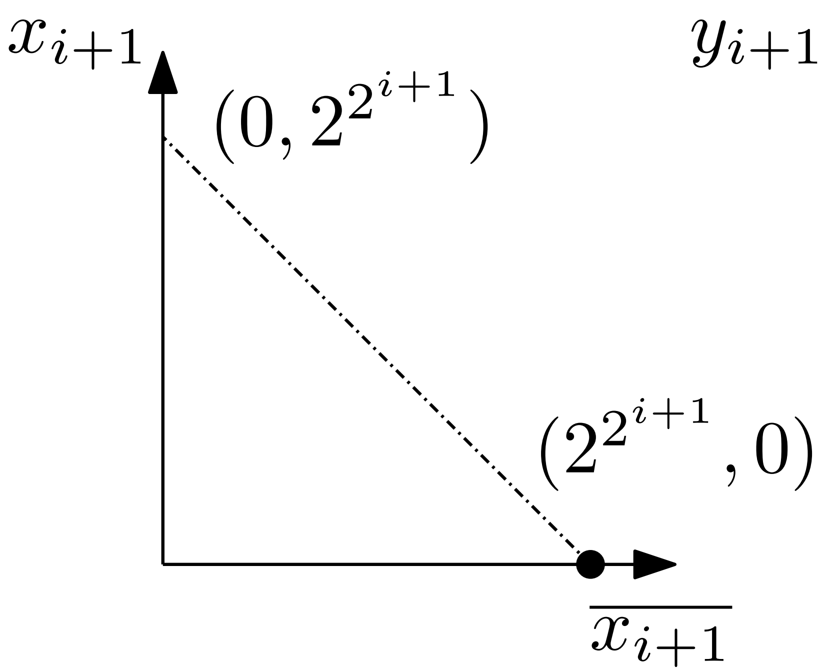
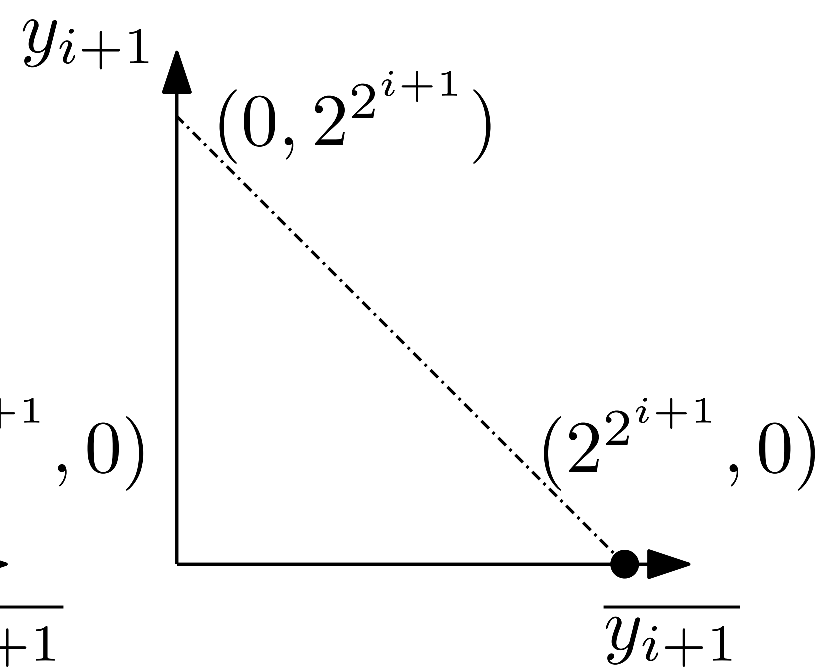


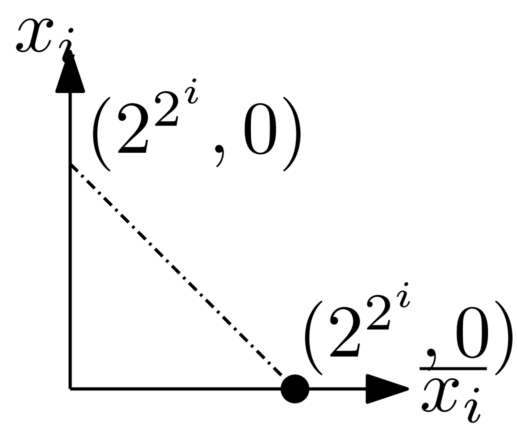


Back to the predecessor algorithm

We define a family $\{(G_i, \langle q_i, 0, \ldots, 0\rangle)\}_{i\in\mathbb{N}_0}$ of VASS+G_i-state for which the sequence $Z_1, Z_2, \ldots$ given by

$$Z_1 = \{\langle q_i, 0, \ldots, 0\rangle\}$$
$$Z_{j+1} = \min\Big(\{\langle q_i, 0, \ldots, 0\rangle\} \cup minpre(Z_j)\Big)$$

is such that:



- $\dagger_i \geq 2^{2^i}$
- $|Z_{\dagger_i}| \geq 2^{2^i}$
- the highest number in $Z_{\dagger_i}$ is at least $2^{2^{\Omega(i)}}$

Conclusions

• The predecessor computation has been showed to be optimal w.r.t. the complexity of the coverability problem

- ▶ Easily derived from the complexity proof

• Rather surprising contrast with the forward algorithm (Karp and Miller) that is non-recursive primitive

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