Efficient Bounded Reachability Computation for Rectangular Automata

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RP 2011

Outline

1. Reachability computation for rectangular automata
2. Compute reachable sets efficiently
3. Comparison with PHAVer
4. Future work

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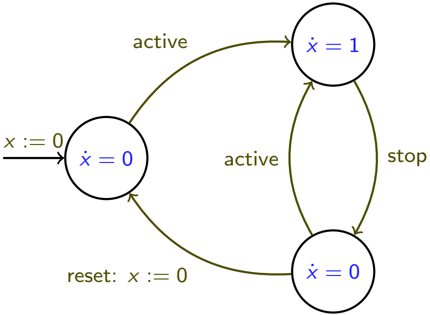

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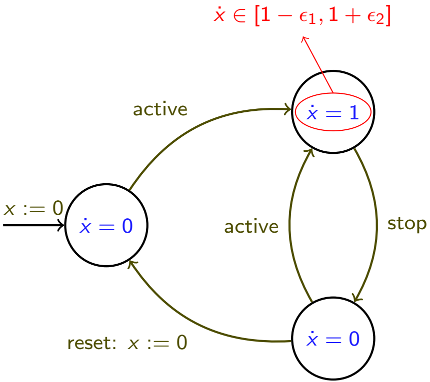

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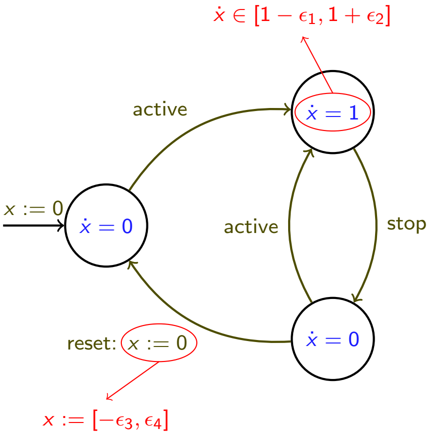
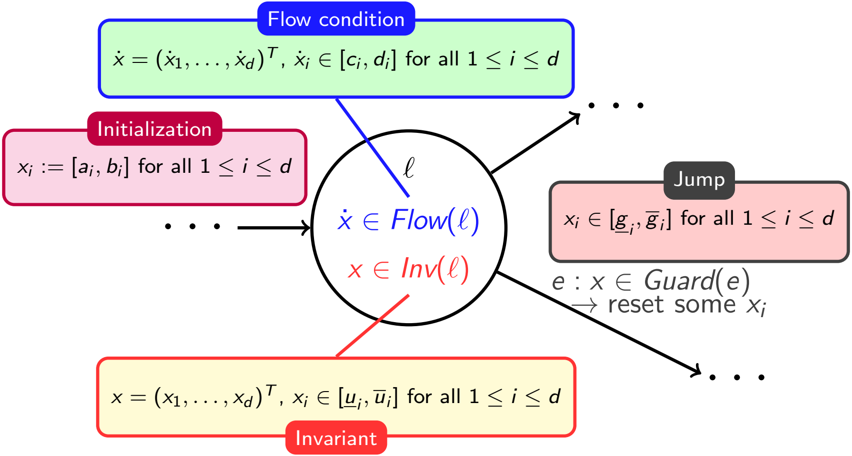
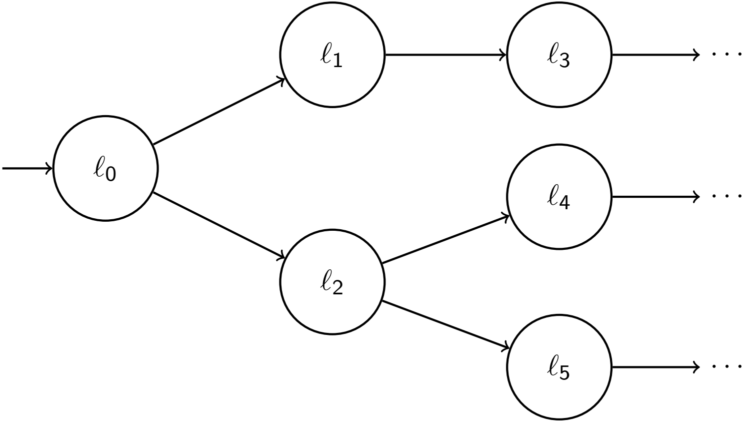


Rectangular automata



Henzinger et al. What's Decidable about Hybrid Automata? In JCSS (1998)

Bounded reachability computation



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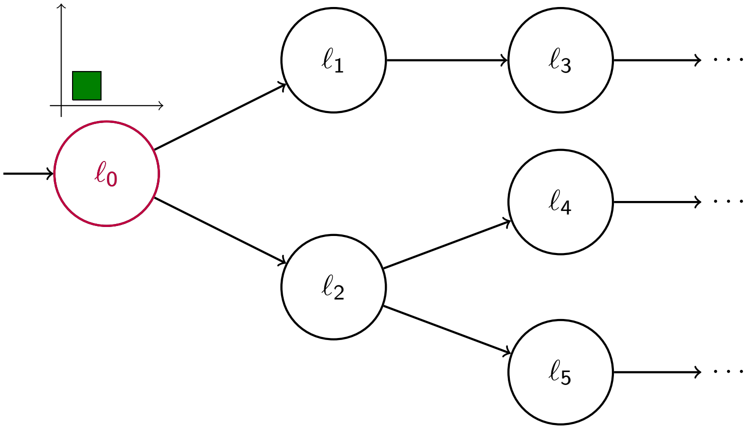

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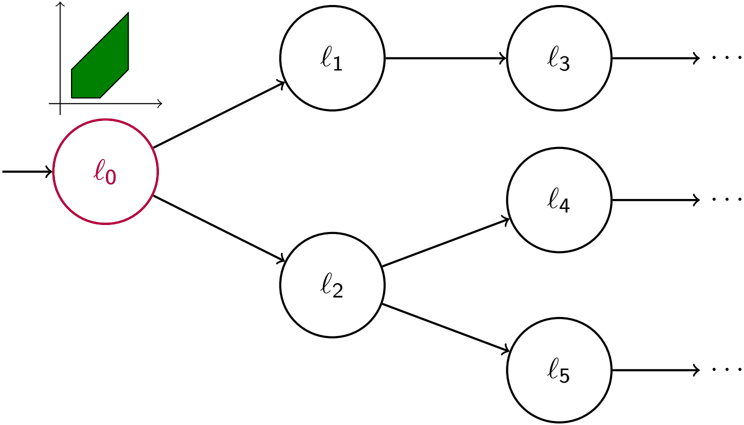

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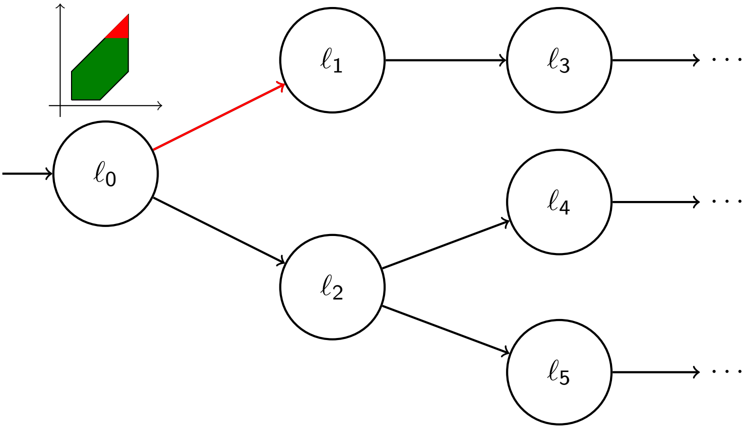

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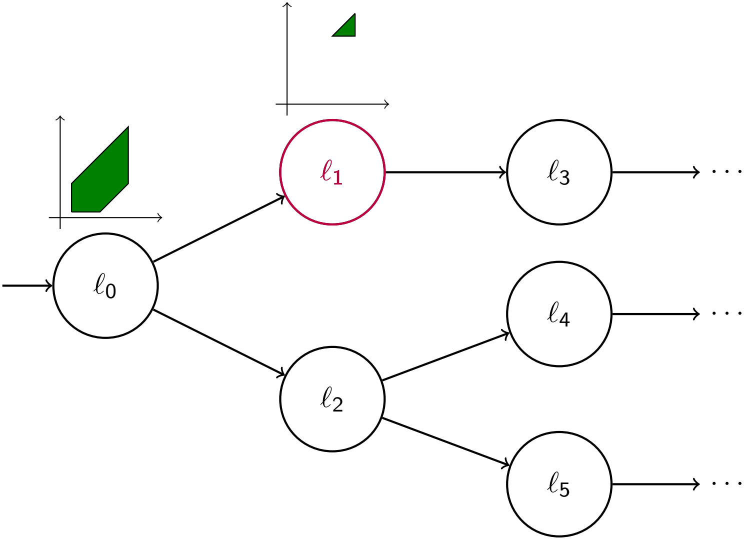

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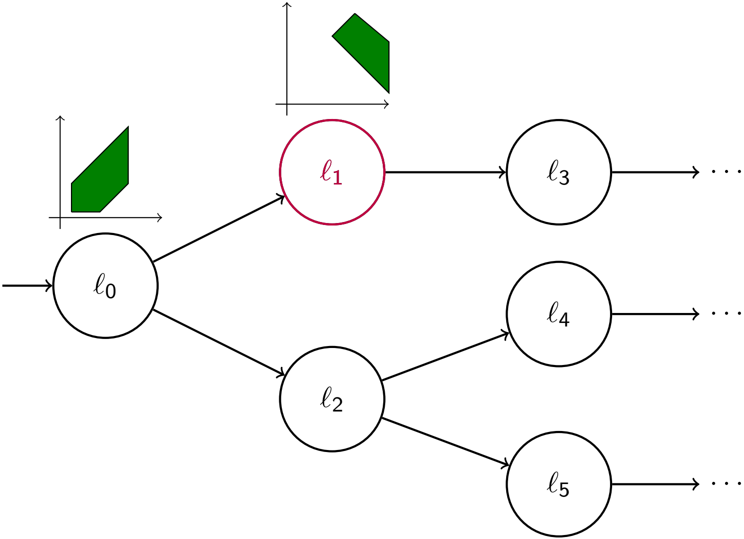

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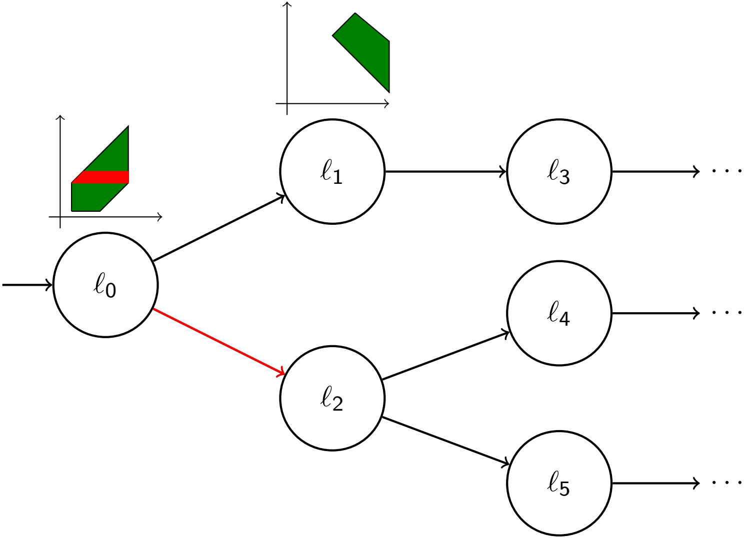

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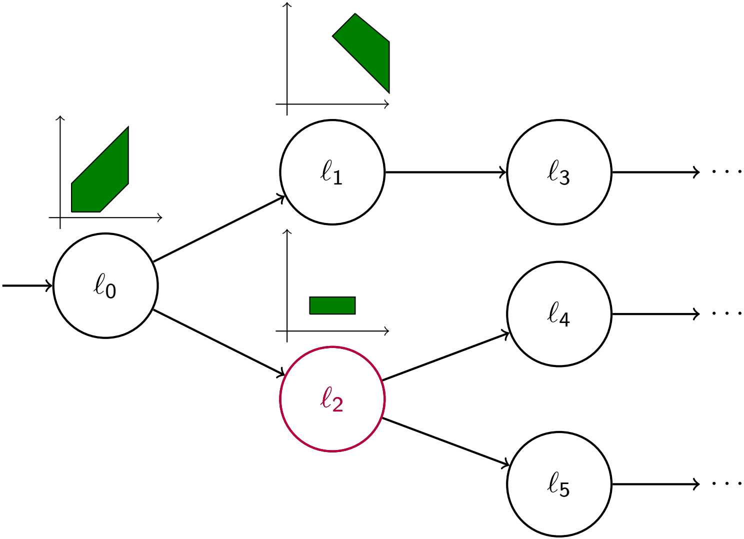

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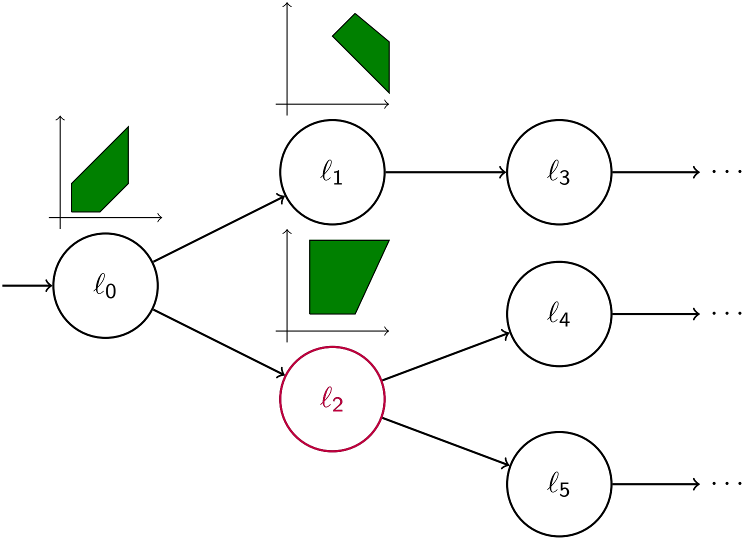

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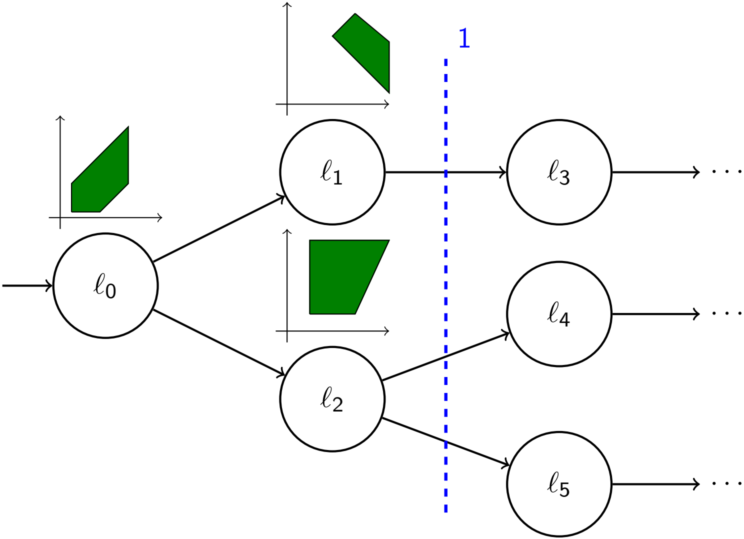

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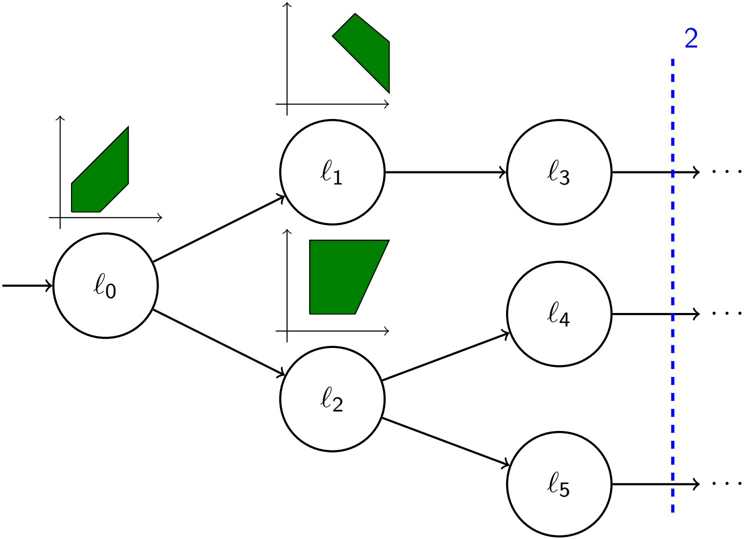

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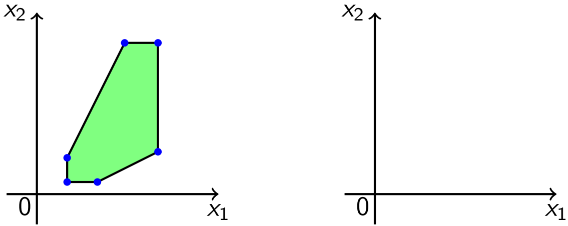

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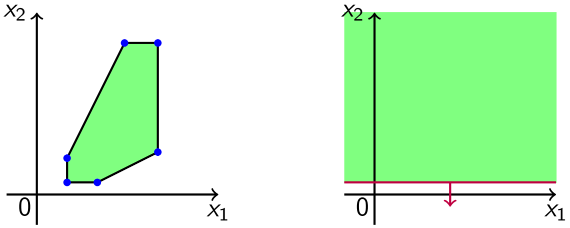

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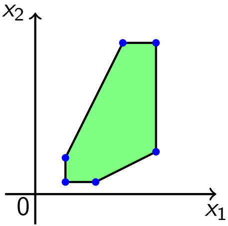

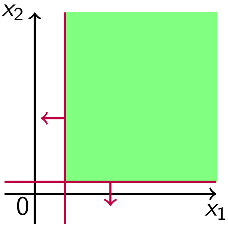

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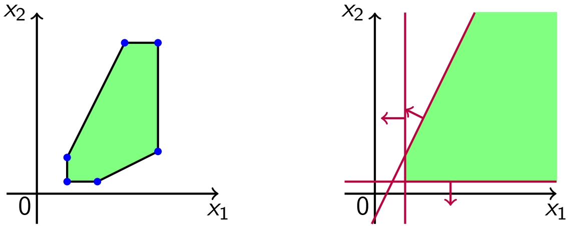

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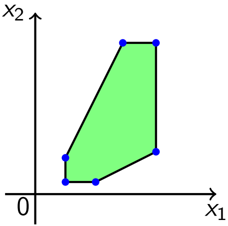

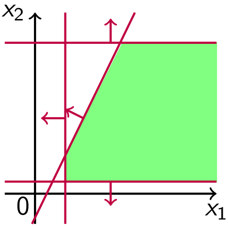

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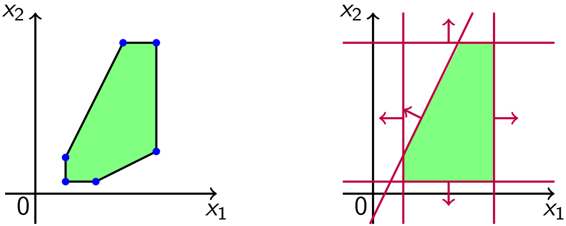

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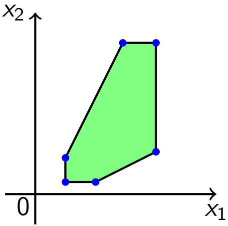

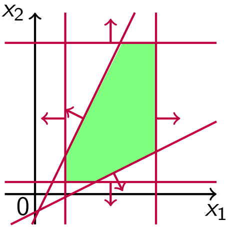

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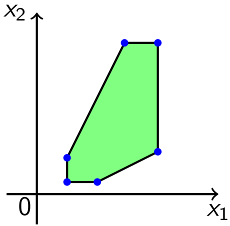

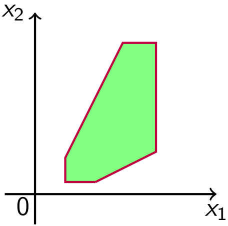

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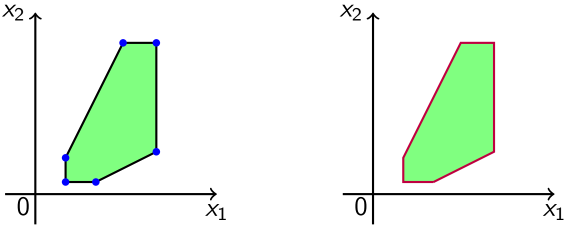

- If $P : \mathcal{L}_P$ and $Q : \mathcal{L}_Q$, then $P \cap Q : \ \mathcal{L}_P \cup \mathcal{L}_Q$.

Facets and constraints

If $P \subseteq \mathbb{R}^d$ and $dim(P) = d'$, then

- facets: $(d' - 1)$-faces, vertices: 0-faces;
- there are $NF(P) + 2(d - d')$ constraints needed to define P where $NF(P)$ is the number of P's facets.

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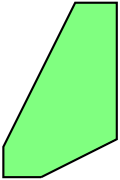

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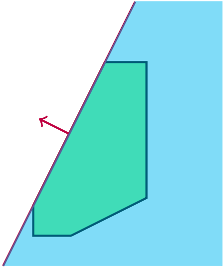

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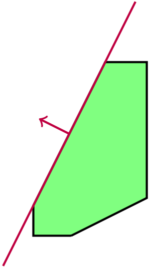

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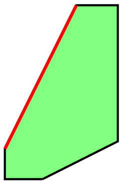

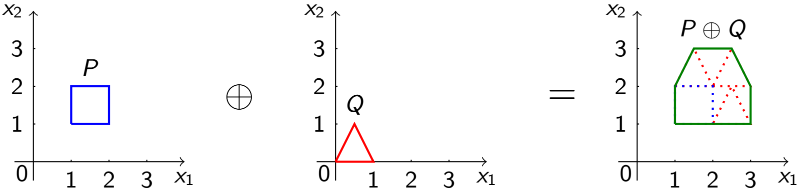

$$P \oplus Q = \{p + q \mid p \in P \text{ and } q \in Q\}$$

Reachable sets under flow transitions

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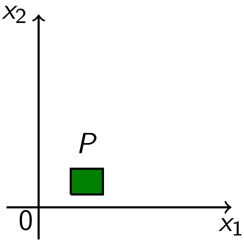

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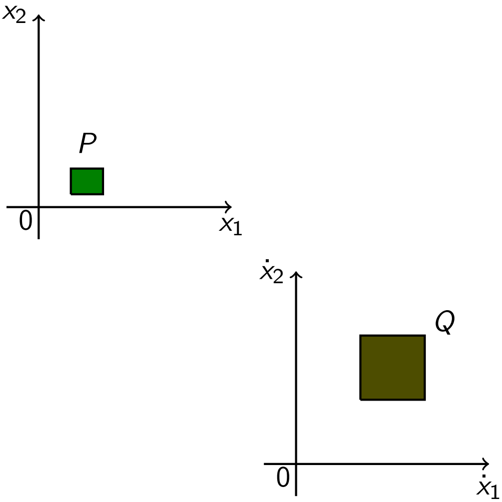

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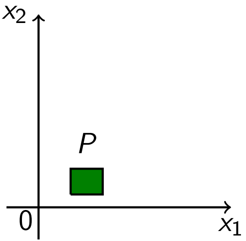

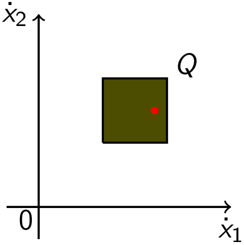

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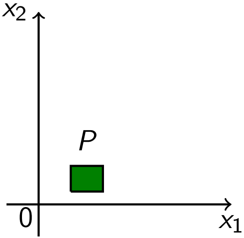

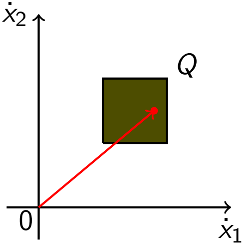

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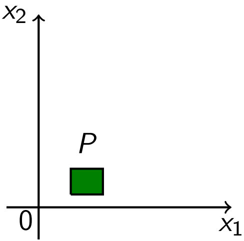

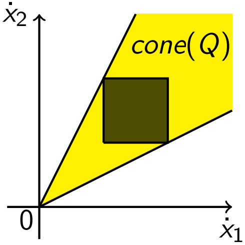

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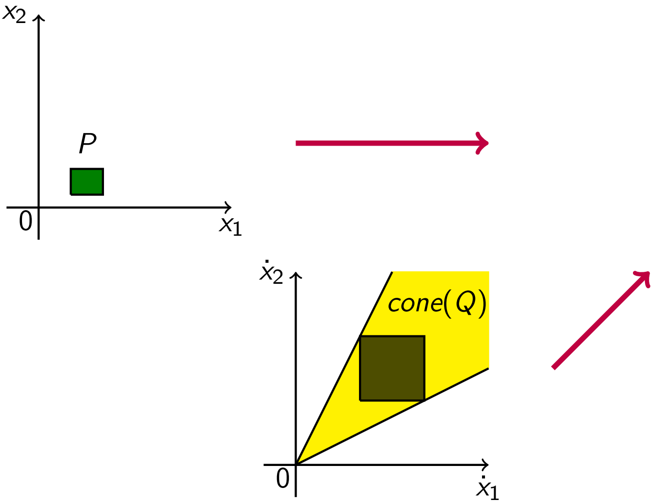

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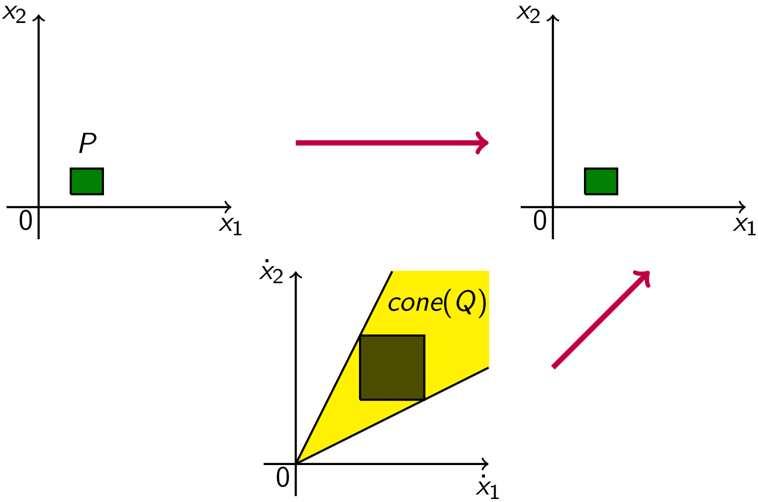

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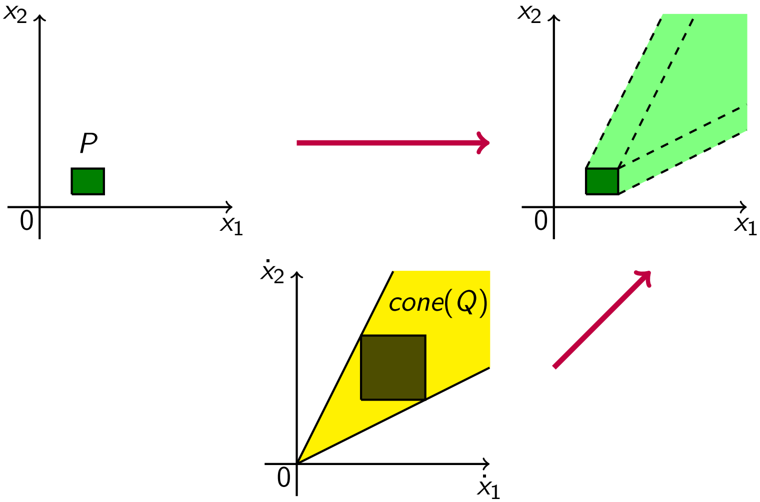

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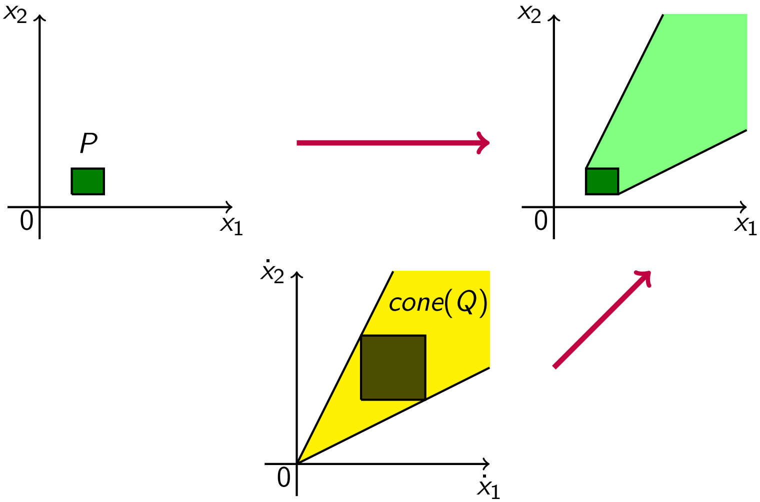

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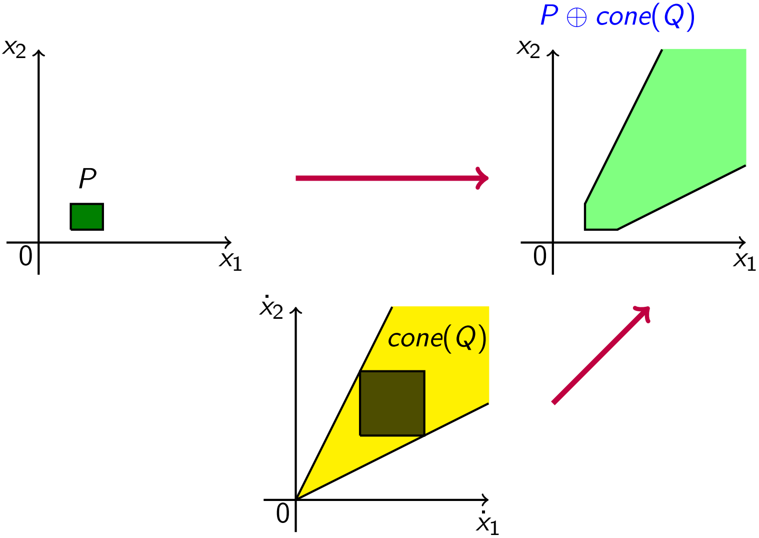

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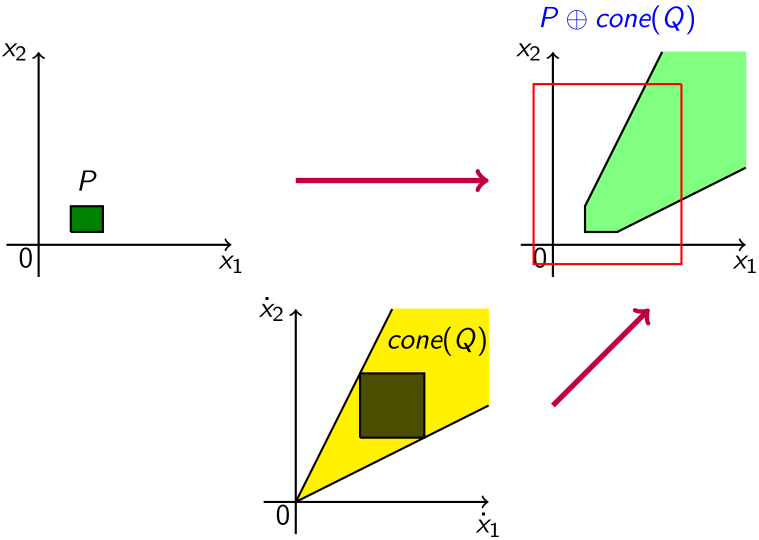

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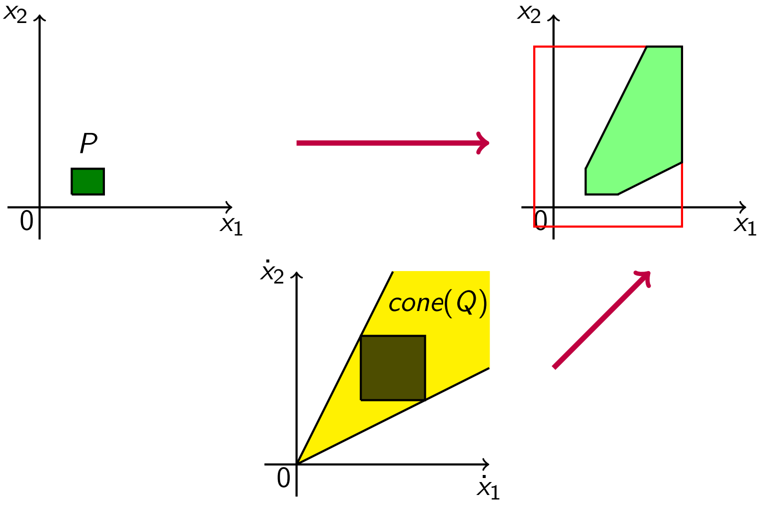

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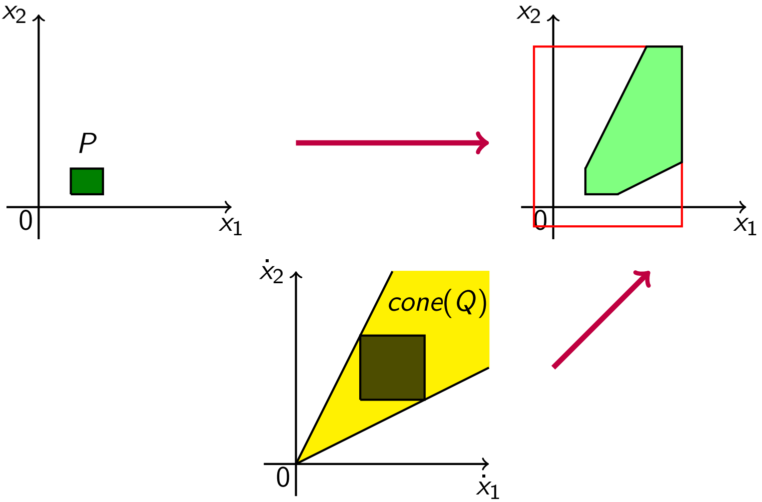
$$R_\ell(P) = (P \oplus cone(Q)) \cap Inv(\ell)$$



Classical method for computing $R_\ell(P)$

Henzinger et al. HYTECH: A Model Checker for Hybrid Systems. CAV'97
Frehse. PHAVer: Algorithmic Verification of Hybrid Systems Past HyTech. HSCC'05

Classical method for computing $R_\ell(P)$

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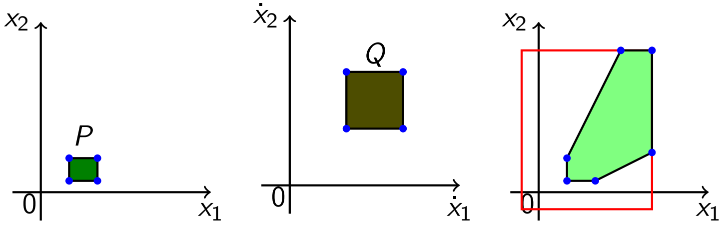

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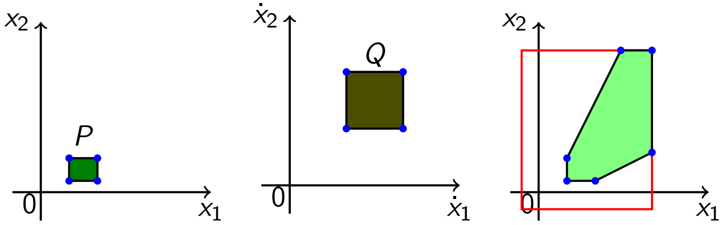

- Used by HyTech and PHAVer.

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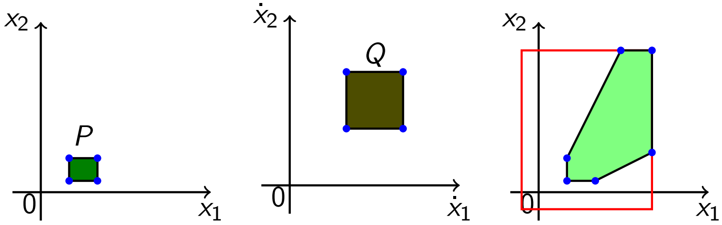

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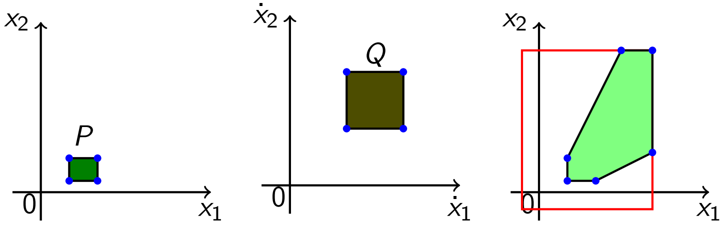

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- Disadvantages:
  1. $O(2^d)$ many vertices for each flow condition;

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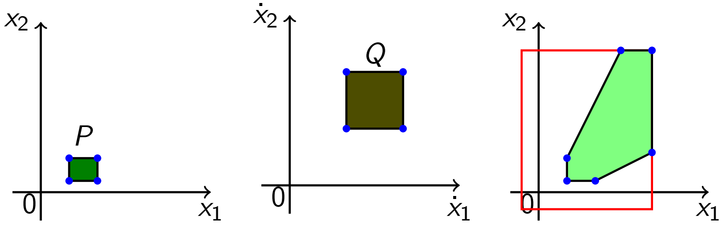

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- Disadvantages:
  1. $O(2^d)$ many vertices for each flow condition;
  2. intersection with an invariant could generate a large number of vertices.

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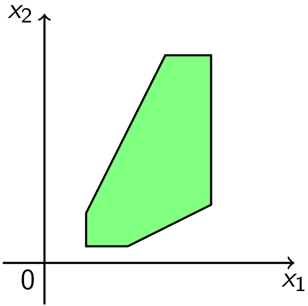
Reachable sets under jumps



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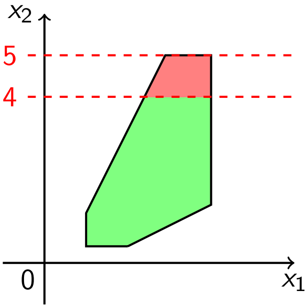

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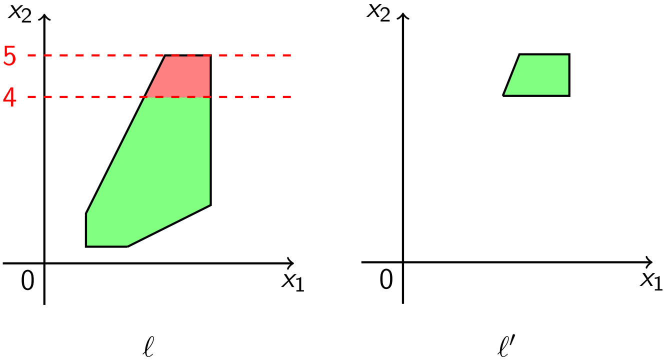

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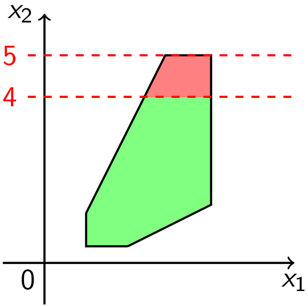
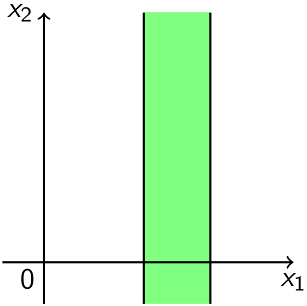


ℓ



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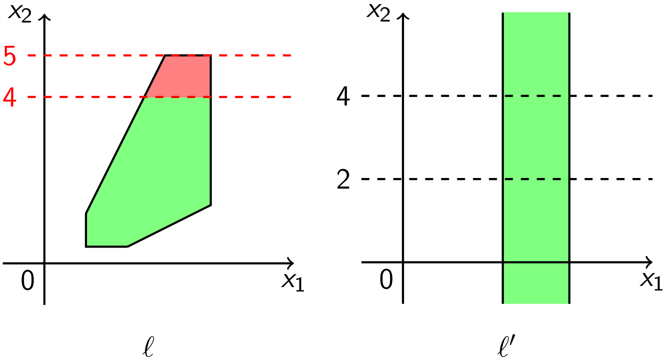

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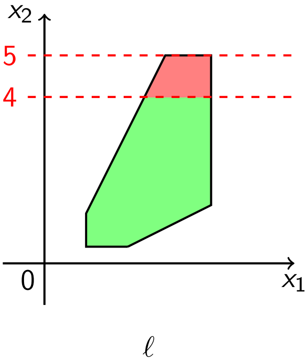

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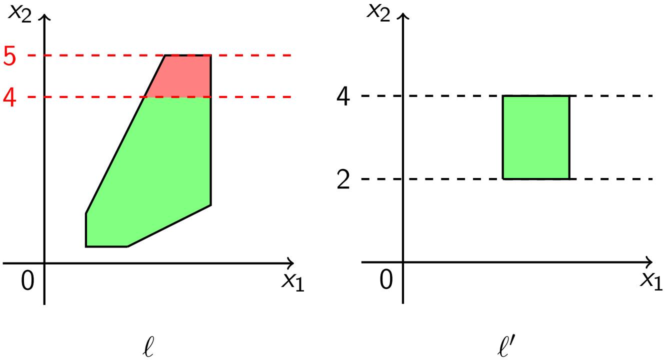

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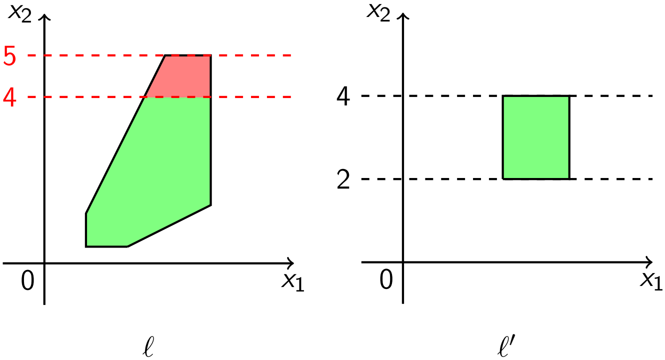

- Computed via projection and Minkowski sum.
- At least $O(2^d)$ many vertices need to be handle.

Our goal

- The reachable set computation under a flow condition is polynomial in d.
- The bounded reachability computation is cheap.

Outline

1. Reachability computation for rectangular automata
2. Compute reachable sets efficiently
3. Comparison with PHAVer
4. Future work

Properties of the facets of R

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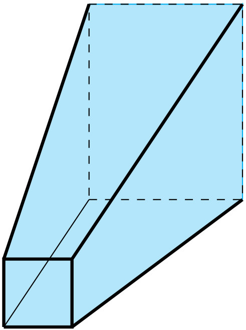

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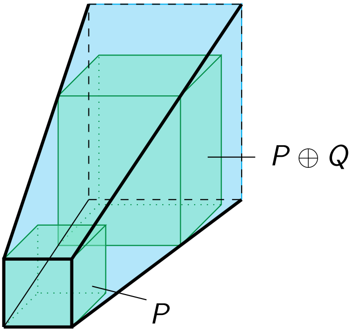

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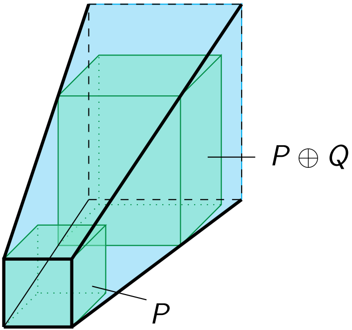

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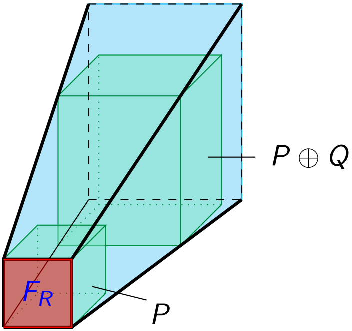

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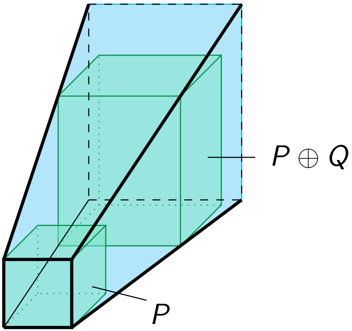

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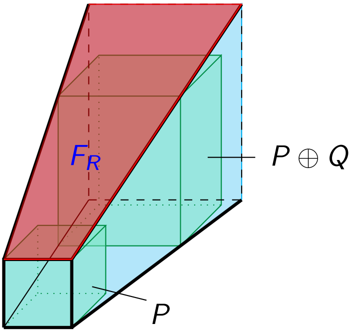

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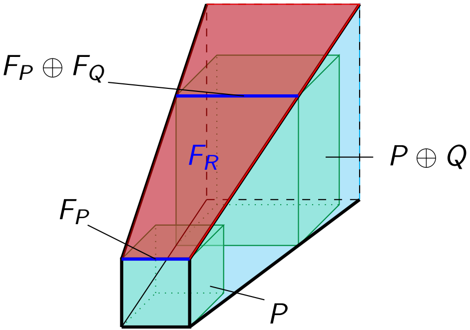

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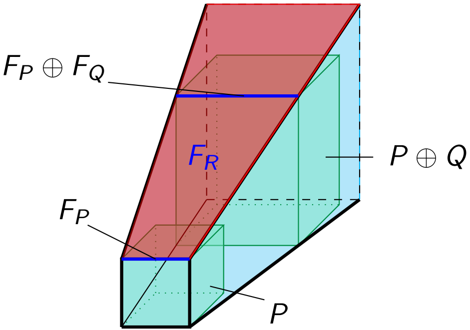
  $F_P \oplus F_Q$ is at least $(d-2)$-dimensional.



How to compute the constraints for R

Assume $P : \mathcal{L}_P$ and $P \oplus Q : \mathcal{L}_{P \oplus Q}$ are d-dimensional.

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3. For every two constraints $L_i, L_j \in \mathcal{L}_{P \oplus Q}$, compute $L_{i,j}$.
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Complexity: $O(|\mathcal{L}_P| + |\mathcal{L}_{P\oplus Q}| + |\mathcal{L}_{P\oplus Q}|^2)$ linear programs need to be solved.

Check the validity of a constraint

Polyhedron $S : \mathcal{L}_S$.

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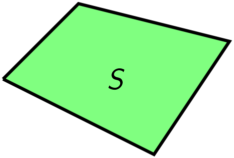

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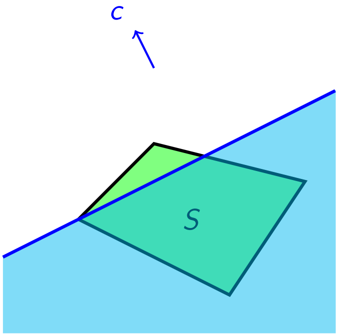

$L : c^T x \leq z'$

$z' < z$

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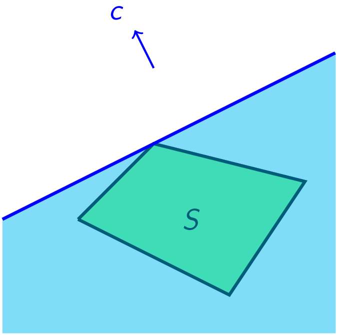

$L : c^T x \leq z'$

$z' = z$

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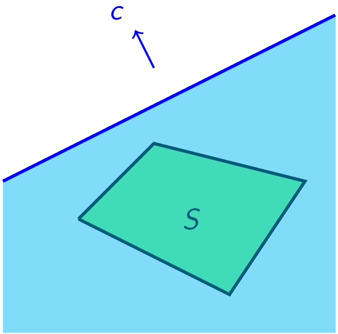

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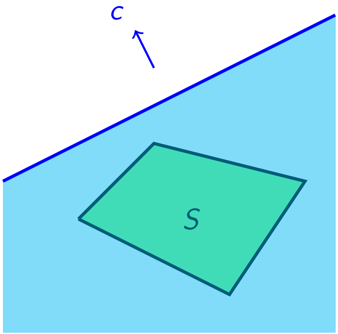

$L : c^T x \leq z'$
$z' > z$

$$L \text{ is valid for } S \text{ iff } \rho_S(c) \leq z',$$

where $\rho_S(c) = \sup c^T x \quad s.t. \quad x$ satisfies all $L_S \in \mathcal{L}_S$.

The validity of a constraint for R

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For any vector $c \in \mathbb{R}^d$ we have that

$$\rho_R(c) = \rho_{P \oplus cone(Q)}(c) = \sup_{\lambda \geq 0}(\rho_P(c) + \lambda \cdot \rho_Q(c))$$

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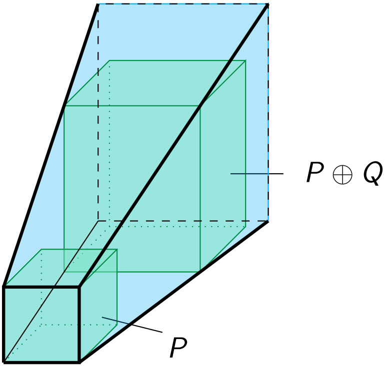
A constraint $L : c^T x \leq z$ is valid for R iff

$$\rho_P(c) \leq z \text{ and } \rho_Q(c) \leq 0.$$

An example of $L_{i,j}$



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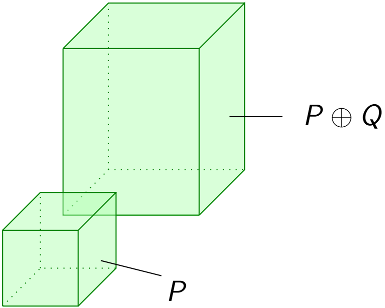

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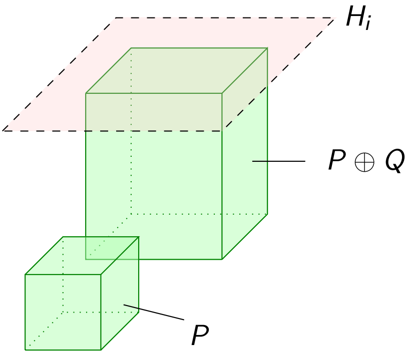

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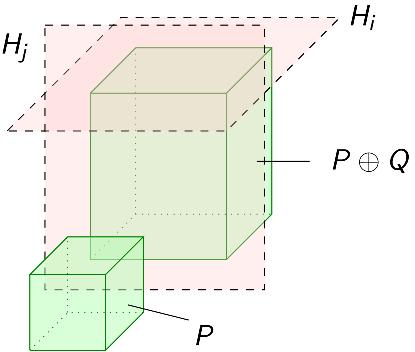

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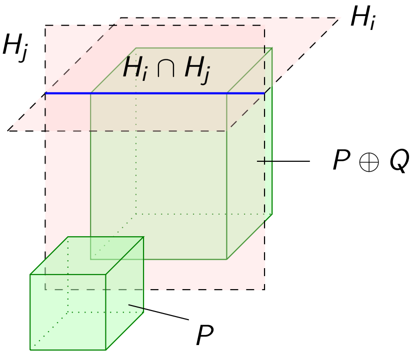

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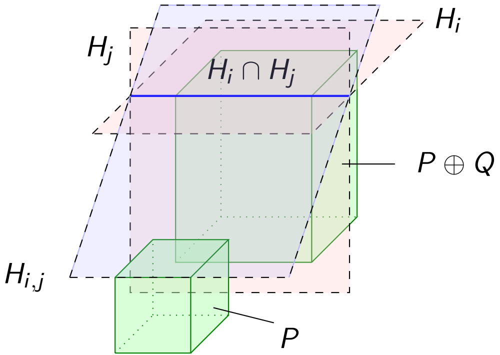

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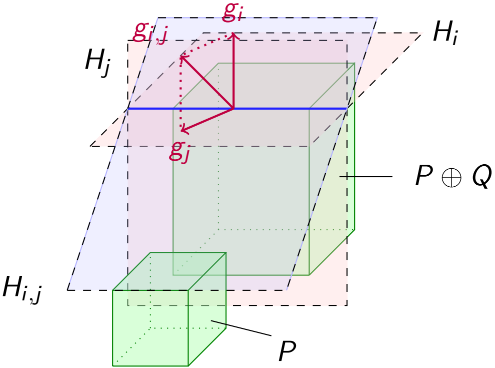
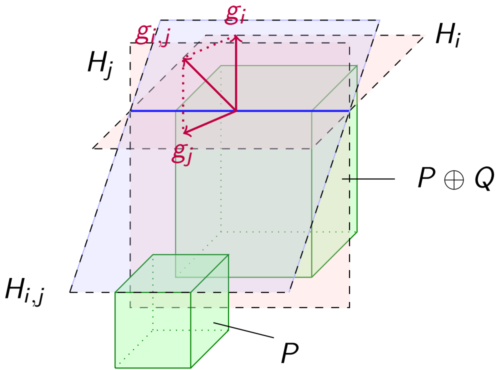


$g_{i,j} = \alpha g_i + \beta g_j$ where $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$.

An example of $L_{i,j}$



$g_{i,j} = \alpha g_i + \beta g_j$ where $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$.

$H_{i,j} : c^T x = z$ can be found by linear programming and $L_{i,j} : c^T x \leq z$.

Compute the reachable set after a jump

Assume $\mathcal{L}$ defines the reachable set under a flow condition.

1. Eliminate all reset variables from the constraints in $\mathcal{L}$ by Fourier-Motzkin elimination.
2. Add the constraints $x_i \leq b, -x_i \leq -a$ into the new constraint set if there is a reset $x_i := [a, b]$.

Complexity of the computation

- The set of bounded executions along the location sequence:

$$\ell_0 \xrightarrow{e_1} \ell_1 \xrightarrow{e_2} \cdots \xrightarrow{e_k} \ell_k$$

- The corresponding computation sequence:

$$R_{\ell_0}(X_0) \xrightarrow{e_1} R_{\ell_1}(X_1) \xrightarrow{e_2} \cdots \xrightarrow{e_k} R_{\ell_k}(X_k)$$

where $X_j = R_{e_j}(R_{\ell_{j-1}}(X_{j-1}) \cap Guard(e_j) \cap Inv(\ell_{j-1}))$ for $1 \leq j \leq k$.

Complexity of the computation

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- The set X_j can be expressed by

$$\bigcup_{a_{j-1} \leq \lambda_{j-1} \leq b_{j-1}} \cdots \bigcup_{a_0 \leq \lambda_0 \leq b_0} R_{e_j}((\cdots R_{e_1}((X_0 \oplus \lambda_0 B_0) \cap G_0) \cdots \oplus \lambda_{j-1} B_{j-1}) \cap G_{j-1})$$

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- The number of the facets of X_j is bounded by

$$\mathcal{F}_j = \sum_{\max(d-j-1,0)\le d'\le d-1} \binom{j}{d-d'-1} 2^{d-d'} \binom{d}{d'}$$

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- $\mathcal{F}_j$ is polynomial in $\mathcal{F}_{j-1}$.
- If j is fixed, then $\mathcal{F}_j$ is polynomial in d when d is large enough.

Main result

Theorem

The computational complexity of the reachable set with a bounded number of jumps is polynomial in d if the bound is viewed as a constant and d is large enough.

Outline

1. Reachability computation for rectangular automata
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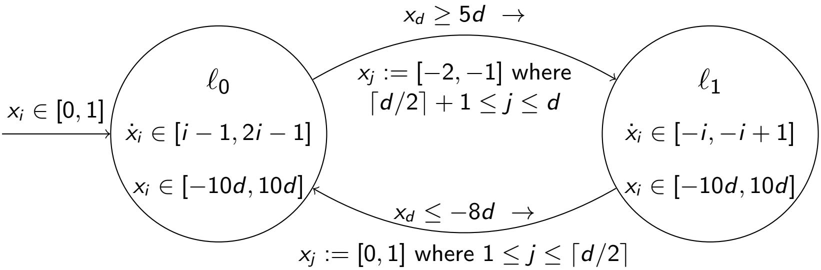
The scalable model



The experimental results

Dim	Jmp	PHAVer		Our method (on MATLAB)				
		Mem	Time	Mem	Time	ToLP	LPs	Cons
5	2	9.9	0.81	< 10	2.36	2.20	1837	81
6	2	48.1	21.69	< 10	4.96	4.68	3127	112
7	2	235.7	529.01	< 10	15.95	15.28	7214	163
8	2	-	-	< 10	27.42	26.48	10517	209
9	2	-	-	< 10	107.99	105.59	23639	287
10	2	-	-	< 10	218.66	215.45	32252	354
5	4	10.2	1.51	< 10	4.82	4.50	3734	167
6	4	51.1	35.52	< 10	11.25	10.64	7307	240
7	4	248.1	1191.64	< 10	32.93	31.60	16101	352
8	4	-	-	< 10	72.04	69.81	27375	466
9	4	-	-	< 10	240.51	235.61	64863	641
10	4	-	-	< 10	543.05	535.77	86633	816

Platform: Intel I7 2.8 GHz CPU, 4GB memory, Linux

Outline

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Future work

- Bounded reachability computation for linear hybrid automata.
- Synthesis of switching controllers for linear hybrid automata.
- Approximative reachability computation for nonlinear systems.